A MATHEMATICAL MODEL FOR FLEXURAL PEELING
OF SIDE-PLATED GLUED BEAMS

by

N T Nguyen
D J Oehlers

Research Report No. R143
February 1997

ISBN No. 0 86396 437 0
A MATHEMATICAL MODEL FOR FLEXURAL PEELING OF SIDE-PLATED GLUED BEAMS

N T Nguyen      D J Oehlers
Research Fellow  Senior Lecturer

Department of Civil & Environmental Engineering
The University of Adelaide

ABSTRACT: This work on gluing steel plates to the sides of existing reinforced concrete beams is a continuation of earlier research on gluing steel plates to the soffits of concrete beams (Ref. 1). A generic mathematical model has been developed for simulating the debonding of side plates due to flexural peeling and this model has been calibrated using six full-scale side-plated glued beam tests.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>i</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>ii</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2. Forces Acting on Side-plates</td>
<td>1</td>
</tr>
<tr>
<td>3. Mathematical model for flexural peeling</td>
<td>4</td>
</tr>
<tr>
<td>3.1 Transmission of Flexural Moment $M_p$</td>
<td>4</td>
</tr>
<tr>
<td>3.2 Transmission of Axial Force $F_p$</td>
<td>5</td>
</tr>
<tr>
<td>3.2.1 Direct stress</td>
<td>5</td>
</tr>
<tr>
<td>3.2.2 Mean shear stress</td>
<td>7</td>
</tr>
<tr>
<td>3.3 Interaction Between $(\tau_{sh})<em>{m}$, $\tau</em>{max}$ and $f_{\bar{u}}$</td>
<td>7</td>
</tr>
<tr>
<td>3.4 Interaction Between $(\tau_{sh})<em>{e}$, $\tau</em>{max}$ and $f_{a}$</td>
<td>9</td>
</tr>
<tr>
<td>4. Calibration of the Model</td>
<td>10</td>
</tr>
<tr>
<td>5. Conclusions</td>
<td>12</td>
</tr>
<tr>
<td>6. Acknowledgement</td>
<td>12</td>
</tr>
<tr>
<td>7. References</td>
<td>12</td>
</tr>
<tr>
<td>8. Appendix: Notation</td>
<td>13</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

For the last two decades, gluing steel plates to damaged reinforced concrete beams and slabs in order to strengthen and stiffen them has become quite common practice in the construction industry. Although this plating technique has a number of advantages such as being simple, rapid and can be applied whilst the structure is still in use, premature failure due to separation between the plate and concrete can occur that can considerably reduce the effectiveness of this technique. The main objective of this work is to develop a mathematical model for preventing debonding due to flexural forces which is one of several possible mechanisms of failure.

2. FORCES ACTING ON SIDE-PLATES

Let us consider a RC-beam that is subjected to a two point load and in which the side plates are terminated within a constant moment region as shown in Fig. 1.

Fig. 1. Beam Geometry: (a) Test setup; (b) Beam cross-section

Consider a cross-section of the plated beam at a distance equal to the depth of the plate $h_p$ from the end of the side plate as shown in Fig. 2(b). In this analysis, it is assumed that the behaviour of the composite beam is linear elastic as we are dealing with the debonding of real structures where the plates are terminated a long way from the position of maximum moment.

From Fig. 2(b), the mean strain in the plate is

$$\varepsilon_p = h_{p,cm} \phi$$ (1)
and, therefore, the axial force acting on the side plate can be expressed as

\[ F_p = A_p (E\varepsilon)_p = (EA)_p h_{p,cmp} \phi \]  \hspace{1cm} (2)

As

\[ \phi = \frac{M_{cmp}}{(EI)_{cmp}} \]  \hspace{1cm} (3)

substituting Eq. (3) into Eq. (2) gives

\[ F_p = \frac{(EA)_p h_{p,cmp} M_{cmp}}{(EI)_{cmp}} \]  \hspace{1cm} (4)

Fig. 2. Strain profile of a side-plated RC-beam in constant moment region

Now let us consider the forces acting on each element as shown in the free body diagram in Fig. 3. As there are two side plates, the axial force in the reinforced concrete element will be twice the axial force acting on each side-plate that is

\[ F_{RC} = 2F_p \]  \hspace{1cm} (5)

From equilibrium of moments, we get

\[ M_{cmp} = M_{RC} + 2M_p + 2F_p h_{p,RC} \]  \hspace{1cm} (6)

and from the compatibility of the curvatures of the plate and concrete elements
\[ \phi = \frac{M_{RC}}{(EI)_{RC}} = \frac{M_p}{(EI)_p} = \frac{M_{RC} + 2M_p}{(EI)_{RC} + 2(EI)_p} \] (7)

From Eqs. (6) and (7), a new relationship for \( \phi \) can be derived as

\[ \phi = \frac{M_{cmp} - 2F_p h_{p,RC}}{(EI)_{RC} + 2(EI)_p} \] (8)

and substituting Eq. (2) into Eq. (8) gives

\[ \phi = \frac{M_{cmp} - 2\phi(EA)_p h_{p,RC} h_{p,cmp}}{(EI)_{RC} + 2(EI)_p} \] (9)

and rearranging Eq. (9) and solving for \( \phi \) gives

\[ \phi = \frac{M_{cmp}}{(EI)_{RC} + 2(EI)_p + 2(EA)_p h_{p,RC} h_{p,cmp}} \] (10)

As the curvature \( \phi \) for the applied moment \( M_{cmp} \) is known, this can be used to determine the moment in the plate \( M_p = \phi (EI)_p \) (and the moment in the concrete element \( M_{RC} = \phi (EI)_{RC} \)). Hence from Eq. (6), the axial force in the plate \( F_p \), which can be used to ensure that the plate remains linear elastic.

Fig. 3. Free body diagram of side plates and RC-beam
From Eqs. (3) and (10) we can get the following relationship for \((EI)_{cmp}\) as

\[
(EI)_{cmp} = (EI)_{RC} + 2(EI)_{p} + 2(EA)_{p} h_{p,RC} h_{p,cmp}
\]

(11)

which is correct when all the elements remain isotropic. However, to allow for cracking, the following research will be based on the flexural rigidity of the cracked plated section that assumes that the tensile strength of the concrete is zero. Previous research (Ref. 1) has shown that this rigidity gives the least scatter of results.

3. MATHEMATICAL MODEL FOR FLEXURAL PEEING

3.1 Transmission of Flexural Moment \(M_{p}\)

Let us assume that the ends of the plate of area of \((h_{p} \times h_{p})\) transmits the flexural moment \(M_{p}\) from the RC-beam into the plate as shown shaded in Fig. 4(a). The moment \(M_{p}\) is transmitted into the plate through the shear forces acting at the interface between the concrete and side plate. It will be assumed that premature debonding occurs at the corners of the square bonded area in Fig. 4(a), so that \(M_{p}\) is transmitted by shear in the circular bonded area in Fig. 4(b) which is shown enlarged in Fig. 5.

![Diagram showing transmission of flexural forces](image)

(a) perfectly bonded  (b) allowing for debonding

Figure 4. Transmission of the flexural forces through the ends of the side plates
As a linear elastic analysis is being applied, the shear stress $\tau_h$ at a distance $h$ from the center of the bond area varies linearly and is given by

$$\tau_h = \frac{2h\tau_{\text{max}}}{h_p}$$ (12)

where $\tau_{\text{max}}$ is the maximum shear stress at the circumference. Hence the moment increment due to the shear stress $\tau_h$ is $dM = 2\pi h \tau_h \, dh = 4\pi h^3 \tau_{\text{max}} \, dh / h_p$ and integrating over the bonded area gives

$$M_p = \frac{h_r/2}{0} dM = \frac{\pi . h_p^3 . \tau_{\text{max}}}{16}$$ (13)

which gives the maximum shear stress $\tau_{\text{max}}$ as

$$\tau_{\text{max}} = \frac{16 M_p}{\pi h_p^3}$$ (14)

3.2 Transmission of Axial Force $F_p$

3.2.1 Direct stress

Let us consider how the axial force $F_p$ is transmitted from the RC beam to the side plates as shown in the plan view of the plated beam in Fig. 6.
From the equilibrium of the flexural forces in Fig. 6, \( F_p \left( \frac{t_p}{2} \right) = F_a \ k_1 \ t_p \), therefore,

\[
F_a = \frac{F_p}{2 \ k_1}
\]  

(15)

The stress distribution across the interface is shown adjacent to the top plate in Fig. 6 where \( f_a \) is the maximum tensile stress and where the tensile stress is distributed over the length \( (k_2 \ t_p) \) and depth of plate \( h_p \). As the thickness of the plate \( t_p \) is usually much less than the width of the beam \( b_c \), the distribution of stress must be a function of \( t_p \) as has been shown in finite element analyses (Ref. 1). If we define the shape of the tensile stress distribution as \( s_a \) where the mean tensile stress is \( (s_a f_a) \), then

\[
F_a = (s_a f_a) (k_2 t_p) h_p
\]  

(16)

Substituting Eq. (16) into Eq. (15) gives

\[
f_a = (2k_1k_2s_a)^{-1} f_p = k_a f_p = k_a E_p \varepsilon_p
\]  

(17)

and substituting Eq. (1) into Eq. (17) gives

\[
f_a = k_a E_p \phi h_{p,cm}
\]  

(18)
3.2.2 Mean shear stress

Figure 6 also shows that the axial force in the plate $F_p$ must be equal to the shear force $F_{sh}$ acting on the interface between the concrete and steel elements, that is

$$F_p = F_{sh} \quad (19)$$

Let us assume that $L_{sh}$ is the effective bond length for this shear force $F_{sh}$ over which an average shear stress $(\tau_{sh})_m$ is acting. It is also reasonable to assume that $L_{sh}$ is proportional to the thickness of the plate (ref. 1) so that $L_{sh} = k_{sh}t_p$ as shown in Fig. 6. Hence, the mean shear stress $(\tau_{sh})_m$ can be written as

$$ (\tau_{sh})_m = \frac{F_{sh}}{k_{sh}t_p h_p} \quad (20)$$

From Eqs. (15), (19) and (20), we get

$$ (\tau_{sh})_m = \frac{2F_a k_1}{k_{sh}t_p h_p} \quad (21)$$

and substituting Eq. (16) into Eq. (21) and simplifying gives

$$ (\tau_{sh})_m = \frac{2s_a k_1 k_2 f_a}{k_{sh}} = \frac{f_a}{k_a k_{sh}} \quad (22)$$

It is worth noting that at the plate edge, the shear stress at the edge $(\tau_{sh})_e$ is zero as there is a free surface.

3.3 Interaction Between $(\tau_{sh})_m$, $\tau_{max}$ and $f_a$

Let us now consider the critical point at the interface where $(\tau_{sh})_m$, $\tau_{max}$ and $f_a$ are at their maximum, which occurs at the middle of the edge of the plate end as shown in Fig. 7. If $\tau_R$ is the resultant of $(\tau_{sh})_m$ and $\tau_{max}$ in Fig. 7(b), then we have an element that is subjected to a shear stress $\tau_R$ and tensile stress $f_a$ as shown in Fig. 7(c).
Fig. 7. Debonding stresses

If we assume that the debonding occurs when the principal tensile stress in Fig. 7(c) is equal to the tensile strength of the concrete, then from Mohr’s stress circle

\[ 0.5f_a + \sqrt{\tau_R^2 + (0.5f_a)^2} = f_t \]  

(23)

The parameter \( \sqrt{\tau_R^2 + (0.5f_a)^2} \) in Eq. (23) can be written as \( k_R \tau_R + 0.5f_a \) where \( k_R = f(\tau_R, f_a) \). If we assume as a first approximation that \( k_R \) is constant, then Eq. (23) becomes

\[ f_a + k_R\tau_R = f_t \]

(24)

As both the shear stress components \( (\tau_{sh})_m \) and \( \tau_{max} \) are acting on the same plane as in Fig. 7(b), then a relationship for its resultant can be written as

\[ \tau_R = \sqrt{\tau_{sh}^2 + \tau_{max}^2} = k_{sh}^* (\tau_{sh})_m + \tau_{max} \]

(25)

where \( k_{sh}^* = f((\tau_{sh})_m, \tau_{max}) \).

Similarly, we can assume \( k_{sh}^* \) is constant as a first approximation and substituting Eq. (25) into Eq. (24) gives

\[ f_a + k_R[k_{sh}^*(\tau_{sh})_m + \tau_{max}] = f_t \]

(26)

and substituting Eq. (22) into Eq. (26) and simplifying gives

\[ k_{sh}^* f_a + k_R \tau_{max} = f_t \]

(27)
where \( k^* \) is given by \( \frac{k_Rk^*_{sh}}{k_a{k^*_{sh}}} \).

Now let us derive Eq. (27) further, in terms of the applied load, geometric and material properties of the plated beam. Substituting Eqs. (15) and (20) into Eq. (27) and noting that \( M_p = \phi (EI) \) and \( f_a = f_i \) gives

\[
k_a k^* + \frac{16k_RI_p}{\pi h_{p,cmp}^3} = \frac{f_i}{\phi E_p h_{p,cmp}}
\]

(28)

Furthermore, substituting Eq. (3) into Eq. (28) and \( I_p = t_p h_p^3 / 12 \), \( k_A = k_a k^* \) and \( k_B = 4k_R / (3\pi) \) gives

\[
k_A + k_B \left( \frac{t_p}{h_{p,cmp}} \right) = \frac{f_i (EI)_{cmp}}{M_{cmp} E_p h_{p,cmp}}
\]

(29)

which can be written as the following linear variation

\[
k_A + k_B X = Y
\]

(30)

where the variables \( X = \frac{t_p}{h_{p,cmp}} \) and \( Y = \frac{f_i (EI)_{cmp}}{M_{cmp} E_p h_{p,cmp}} = \frac{f_i}{E_p \epsilon_p} \) are both dimensionless variables. Eq. (30) is a generic form of the mathematical model for flexural peeling of side plated beam.

3.4 Interaction Between \( \tau_{sh}, \tau_{max} \) and \( f_a \)

Let us now consider that at the critical point at the edge of the interface, the edge shear stress is zero whilst \( \tau_{max} \) and \( f_a \) are at their maximum. In this case, the mean shear stress in Fig. 7(b) becomes \( \tau_{max} = (\tau_{sh})_e = 0 \) and resultant shear stress \( \tau_R \) in Fig. 7(c) becomes \( \tau_{max} \). Hence, Eq. (24) can be rewritten as

\[
f_a + k_R \tau_{max} = f_i
\]

(31)

Using similar modifications as in previous sections, Eq. (31) can be rewritten as

\[
k_A^# + k_B \left( \frac{t_p}{h_{p,cmp}} \right) = \frac{f_i (EI)_{cmp}}{M_{cmp} E_p h_{p,cmp}}
\]

(32)
where \( k_A^* = k_a \) and \( k_B = 4k_R / (3\pi) \).

It can be seen that Eq. (32) has similar form as Eq. (29) except for the intercept \( k_A^* \). This means that this equation leads to the same linear variation of the dimensionless variables \( X \) and \( Y \) as described in Eq. (30). Hence, it can be concluded that the magnitude of the mean shear stress \( (\tau_{shh})_m \) does not influence the general form of the relationship between the dimensionless variables \( X \) and \( Y \) in the mathematical flexural peeling model.

4. CALIBRATION OF THE MODEL

A set of six simply supported side-plated glued beams were designed using Eq. (30) in order to determine the values of \( k_A \) and \( k_B \) in Eq. (30). Details of this experimental program are reported elsewhere (Ref. 2). In deriving the variables \( X \) and \( Y \) in Eq. (30), \( M_{cmp} \) was the moment \( M_{up} \) at which flexural peeling occurs, \( f_t \) was the Brazilian tensile strength and \( (EI)_{cmp} \) is the flexural rigidity of the cracked plated beam which was calculated by assuming the tensile strength of the concrete was zero. The results of the tests are shown in Fig. 8.

![Fig. 8. Calibration of the mathematical model](image)

A linear regression analysis of the data in Fig. 8 gave the intercept \( k_A = 0.0172 \), the slope \( k_B = 0.1868 \) and the standard deviation \( STDV=0.0063 \). Substituting these values of \( k_A \) and \( k_B \) into Eq. (30) gives the following mean peeling strength
\[(M_{up})_m = \frac{f_t(EI)_{cmp}}{E_p(0.017h_{p,cmp} + 0.187t_p)} \quad (33)\]

It can be seen in Eqs. (30) and (33) that \(M_{up}\) is inversely proportional to the intercept \(k_A\) and hence the upper 95\% confidence limit in Fig. 8 should be used in design. As there are 12 test results and hence 10 degrees of freedom, the 95\% confidence limit occurs at 1.81 standard deviation from the mean. Hence, \(k_A = 0.0172 + 1.81 \times 0.0063 = 0.0286\). The characteristic peeling strength is, therefore,

\[(M_{up})_{ch} = \frac{f_t(EI)_{cmp}}{E_p(0.0286h_{p,cmp} + 0.187t_p)} \quad (34)\]

The predicted ultimate flexural peeling strengths, \((M_{up})_n\) from Eq. (33), are compared with the experimental strengths, \(M_{exp}\), in Fig. 9. As would be expected, the mean is 1 as the same experimental results were used to calibrate the mathematical model. However, it is worth noting that the scatter is fairly small even though a wide range of plate thicknesses (from 6 to 12 mm) and a wide range of plate depths (from 60 to 240 mm) had been used. It is also worth noting that the prediction equation has been validated over a very wide range of plate peeling strains from about 500 to 1200 micro-strains.

![Fig. 9. Comparison between the predicted and ultimate flexural peeling strengths](image-url)
5. CONCLUSIONS

A simple mathematical model has been developed for predicting the ultimate flexural peeling strength of side-plated glued beams which shows good correlation with experimental data. The ultimate flexural peeling strength was found to be directly dependent on the tensile strength of the concrete and the flexural rigidity of the cracked plated section. However, it was found to be inversely proportional to elastic modulus of the side plate, the plate thickness and the distance between neutral axes of the side plate and composite cracked plated section.

6. ACKNOWLEDGMENTS

This work forms part of an ongoing research project between the Universities of New South Wales and Adelaide on "The Upgrading and Repair of Reinforce Concrete Beams using Externally Bonded Steel Plates" which is funded by a large Australian Research Council Grant.

7. REFERENCES


8. APPENDIX: NOTATION

\[
\begin{align*}
\Lambda &= \text{area} \\
b &= \text{width} \\
d &= \text{depth} \\
dh &= \text{length increment} \\
E &= \text{Young modulus} \\
F &= \text{force} \\
f &= \text{material strength} \\
h &= \text{vertical distance} \\
I &= \text{second moment of area} \\
k &= \text{proportional constant}
\end{align*}
\]
L = longitudinal distance
M = moment
N.A = neutral axis
$s_{n\sigma}$ = mean tensile stress at the concrete/plate interface
STDV = standard deviation
t = thickness
X,Y = dimensionless variables
$\varepsilon$ = strain
$\phi$ = curvature
$\tau$ = shear stress

Suffices
a = axial
c = concrete
ch = characteristic
cmp = composite section
e = edge
m = mean
p = plate
R = resultant
RC = reinforced concrete
sh = shear
t = resultant tensile
up = ultimate peeling moment
exp = experimental