BEHAVIOR OF PLATED RC COLUMNS

By

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ABSTRACT: Both steel jacketing and fibre reinforced polymer (FRP) wrapping have been shown to effectively enhance the seismic resistance of circular reinforced concrete columns by confining the concrete. However, jacketing and wrapping are much less effective in confining concrete within rectangular shaped columns. To solve this problem, an alternative procedure has been developed for improving the ductility of rectangular columns by bolting plates to their surfaces. This new composite partial-interaction plating approach does not rely on confinement to improve the ductility but instead relies on the partial-interaction between the plate and the reinforced concrete column. Both numerical and mathematical studies are conducted in this work to investigate the behaviour of plated columns. Practical procedures are developed for design of such plating systems based on target inter-story drift ratios.
# Table of contents

1 INTRODUCTION ........................................................................................................... 1

2 MECHANISM OF COMPOSITE PLATING .................................................................... 3

3 NUMERICAL STUDIES ............................................................................................... 5
  3.1 EFFECT OF COMPOSITE PLATING ............................................................................. 6
  3.2 FACTORS AFFECTING RESPONSE ............................................................................ 10
    3.2.1 P-Δ Effect ........................................................................................................ 12
    3.2.2 Strength Stiffening ............................................................................................ 14
    3.2.3 Yield Strength ................................................................................................... 16
    3.2.4 Conclusions ................................................................................................…… 19
  3.3 SLIP DISTRIBUTIONS ............................................................................................. 20

4 MATHEMATICAL STUDIES ......................................................................................... 21
  4.1 LINEAR ELASTIC PLUS PLASTIC HINGE MODEL ..................................................... 21
  4.2 LINEAR ELASTIC ANALYSIS .................................................................................. 22
    4.2.1 Generic Mathematical Model .......................................................................... 22
    4.2.1.1 Equilibrium and compatibility .................................................................... 23
    4.2.1.2 Governing differential equation .................................................................... 24
    4.2.2 Solution For The Case Of A Cantilever Column ............................................. 25
  4.3 COMPOSITE PARAMETERS ...................................................................................... 28
    4.3.1 Fundamental Parameters Governing Longitudinal Slip .................................. 28
    4.3.2 Parameters Affecting Deformations .................................................................. 31
  4.4 SLIP DISTRIBUTION OF THE CANTILEVER COLUMN .......................................... 33
    4.4.1 Slip Due To Flexural Moment .......................................................................... 35
    4.4.2 Slip Due To Axial Load ..................................................................................... 36
    4.4.3 Slip Due To Boundary Slip .............................................................................. 37

5 DESIGN OF PLATING SYSTEM .................................................................................. 38
  5.1 GENERIC DEFORMATION - SLIP RELATION ......................................................... 38
  5.2 ULTIMATE PLASTIC HINGE ANALYSIS ................................................................ 39
    5.2.1 Slip in Plastic Hinge Region ............................................................................ 39
    5.2.2 Cross-Sectional Forces ................................................................................... 41
    5.2.3 Calculation Of The Plate Strain ........................................................................ 43
  5.3 DISPLACEMENT BASED PLATING DESIGN PROCEDURE ................................... 46
  5.4 EXAMPLE ................................................................................................................ 48

6 SUMMARY .................................................................................................................. 50

7 REFERENCES .............................................................................................................. 51

8 NOTATION ................................................................................................................... 54
1 INTRODUCTION

The major problems that have been identified (Seible, Priestley, Hegemier and Innamorato, 1997) with regard to the seismic behaviour of reinforced concrete (RC) columns are: shear failure associated with the formation of critical diagonal cracks; flexural failure due to lack of strength and/or ductility within the plastic hinge zone; and lap-spool failure of the longitudinal reinforcement.

Until the early 1990s, the two most common methods for retrofitting deficient RC columns were by constructing an additional reinforced concrete jacket or by installing a grout-injected steel jacket. Steel jacketing is generally more effective than concrete jacketing because the latter results in a substantial increase in the cross-sectional area and self-weight of the structure. Both methods are labour-intensive and sometimes difficult to implement on site. In recent years, the technique of strengthening RC columns using fibre reinforced polymer (FRP) composites has been increasingly used to replace steel jacketing (Saadatmanesh et al 1996, Seible et al 1997). The most common form of FRP column retrofitting involves the external wrapping of FRP sheets/straps in order to confine the concrete within the column. This confinement has been found to be effective mainly for columns with circular or elliptical cross-sections.

Confinement of the concrete by the application of external steel-jacketing/FRP-wrapping improves both the shear and flexural behaviours. For example, increased lateral confinement can substantially enhance the concrete’s axial compressive strength and in particular its axial ductility (Ahmad and Shah 1982; Mander et al 1988a, 1988b). This allows larger rotations within the plastic hinge and hence larger transverse drift as is often required in seismic retrofitting. Many studies have been conducted on the compressive strength of steel/FRP confined concrete columns, particularly axially loaded columns (Fardis and Khalili 1981; Karbhari and Eckel 1994; Mirmiran et al 1998; Saaif et al 1999; Toutanji 1999; Rochette and Labossiere 2000; Xiao and Wu 2000; and Zhang et al 2000; Teng et al 2000; Mirmiran and Shahawy 1997; Samaan et al 1998; Spoelstra and Monti 1999). Furthermore, it has now been well established that steel/FRP confinement is much less effective for square or rectangular columns than for circular columns (Mirmiran et al 1998). Hence, wrapping non-circular columns is often not by itself sufficient for seismic retrofitting.

An alternative procedure, which uses composite partial-interaction plating, is developed in this work to improve the ductility of rectangular RC columns. This new technique of composite partial-interaction plating for rectangular RC columns is illustrated in Fig.1.

Steel or FRP plates are bolted to the opposite faces of a rectangular column as in Figs.1(a) and 1(d). Composite partial-interaction plating and steel/FRP wrapping rely on totally different approaches to enhance the ductility. In wrapping, which is very effective for circular columns, the wrap or sheet is discontinuous at the column/beam joint. The wrap completely encircles the column and the ductility enhancement relies on the triaxial confinement of the concrete provided by the wrap. In contrast, for the composite partial-interaction plating, the plate acts continuously across the column/beam joint. Plates are applied to opposite faces of the column and hence it is suitable for rectangular columns because of their flat surfaces. Furthermore, composite plating relies on the partial-interaction between the plate and the column to enhance the ductility.
To study the effect of the partial interaction plating, the cantilever column shown in Fig. 1 is chosen as a typical case. It can represent either a cantilever bridge column or a typical portion of column in sway frames from the point of contra-flexure near column mid-height to the point of maximum bending moment. Therefore, the conclusions drawn from this study can be considered as general. However, the movement of the column is restricted in the direction of the lateral force (direction of F shown in Fig. 1) in this study. Therefore strictly speaking, the conclusions are only applicable to one directional columns and frames which often occur in bridge columns as well as buildings that are stiffened in one direction by strong shear walls. Side plated columns and two directional columns that are plated on all four faces are beyond the scope of this study. It will, however, be the subject of future research.

All the results from this study are based on the following loading sequence. Firstly, the plates are installed to the unloaded column. Secondly, the axial load is applied to the centroid of the top cross-section and is held constant afterwards. Finally, the lateral load on top of the column is applied which varies in magnitude throughout the loading process. It is possible that in practice the axial load (even some lateral load) is applied to the column before the installation of a retrofitting system. However, it was found from our numerical studies that the difference in response between a column that is retrofitted before axial loading and a column that is axially loaded before retrofitting is minimal. Therefore, other loading sequences are not included in this study. Another assumption made throughout this work is that the width of the plate is the same as that of the RC column, for convenience of study.

In this report, the mechanism by which the composite partial-interaction plating enhances the ductility of rectangular RC columns is first described. This is followed by descriptions of both numerical and mathematical studies of the lateral drift behaviour of composite plated columns. The report concludes with a description
of a design procedure that is based on the required lateral displacement capacity for a column.

2 MECHANISM OF COMPOSITE PLATING

The L-shaped plates in Fig.1(a) are used to provide continuity of the plate across the joint so that when the lateral force $F$ is to the left as shown, the restraint at the bottom of the left hand plate acts as fixed as in Fig.1(b). In contrast, the right hand plate can rise as shown in Fig.1(c) because the restraint on the right is comparatively small as indicated in Fig.1(b). Reversal of the force $F$, such as occurs under seismic loading, simply reverses the restraint conditions. Hence, the essence of this novel technique of composite partial-interaction plating columns is that it increases the capacity of the compressive face without significantly increasing the capacity of the tension face. The increase of compressive resistance in the column delays the onset of concrete crushing, thereby increasing the lateral movement capacity. In addition, the moment capacity of the column is not significantly increased due to the limited increase in tension force capacity in the tension plate due to the “flexible” tension connection of the plate to the joint. Both effects, i.e. the increase of lateral deflection capacity and the limitation of tension force, serve to improve the ductility of the column.

The mechanism by which the above plating system improves the ductility of the column can be further illustrated by considering the equilibrium of forces and compatibility of strains at a cross-section of the column, as shown in Fig.2. The tension plate is ignored in this analysis for the reason mentioned above. Also, bending in the plate is ignored in Fig.2 due to its negligible contribution to the moment (EI of plate << EI of RC column).

\[ M = e_r \cdot N_{cr} + e_t \cdot N_{st} \]
\[ N = N_{cr} - N_{st} \]

![Diagram of forces on the cross-section](image)

The following relation for axial loads is obtained

\[ N = N_{plt} + N_{conc} + N_{sc} - N_{st} \]  \hspace{1cm} (2.1)
in which $N_{\text{con}}$ is the axial force applied to the concrete only; $N_{\text{plt}}$ is the axial force on the plate; $N_{\text{sc}}$ and $N_{\text{st}}$ are the axial forces from the compressive and tensile reinforcement bars respectively; and $N$ is the total axial force or external axial load applied to the cross-section. For the case in which both the compressive and tensile reinforcement bars are yielded before the column fails and in which the column is symmetrically reinforced, we have the following relation

$$N_{\text{sc}} = N_{\text{st}}$$  \hspace{1cm} (2.2)

Therefore, Eq. 2.1 becomes

$$N = N_{\text{plt}} + N_{\text{con}}$$  \hspace{1cm} (2.3)

---

**Fig. 3** Strain profile and stress block in the cross-section
The strain distributions at failure are shown in Fig.3(b) for the unplated and in Fig.3(c) for the plated column. It is assumed that the column fails when the strain at the extreme fiber of the compression zone reaches an ultimate value $\varepsilon_{um}$ (see further discussion in Section 5.2.1). From Eq.2.3 and for the case of an RC column without a plate, the total axial load is resisted by the concrete alone as in Fig.3(b). Whereas and in contrast for the columns with a plate, part of the axial load is transmitted to the plate as in Fig.3(c). Therefore, the axial force on the concrete is reduced. Comparing Figs.3(b) and 3(c), it can be seen that when the axial force in the concrete reduces, the compression zone depth $x$ reduces. For a certain ultimate strain $\varepsilon_{um}$ at the compressive face, reduction of the compressive zone depth $x$ means an increase of ultimate curvature of the cross section, from $\kappa_a$ to $\kappa_b$ as shown. Therefore, the curvature capacity of the cross-section is increased due to plating.

From the above analysis, it can be seen that the reason the steel plating increases the deformation capacity of the column is because the plate attracts part of the axial load and, hence, reduces the axial load on the RC column.

Although the above analysis is based on the assumptions associated with Eq.2.2 and the criteria that the column fails when the ultimate compressive strain is achieved in the compressive face, the conclusion that plating the compressive face increases the deformation capacity is general. This is because the steel plate in the compression face generally reduces the axial load in the RC column. Reducing the axial load on a RC column will increase the deformation capacity of the column. The observation that deformation capacity reduces when axial load increases is well documented in the literature from both numerical and experimental studies (Watson et al. 1994; Watson and Park 1994).

3 NUMERICAL STUDIES

In this section, the effect of the composite partial-interaction plating is studied through numerical simulations of column responses. The responses of plated columns are calculated using a computer program “PLTCOL” that was specifically developed for this type of composite members (Wu, Oehler and Griffith 2001). This numerical modelling is based on a segmental layered procedure whereby a cantilever column, such as in Fig.4, is cut into several segments along its length to form a series of independent cross-sections. Each cross-section is further discretized into many sub-sections as shown in Fig.5(a). Each sub-section retains its own stress-strain history record. Mander’s model (Mander et al., 1988a) is used for the stress-strain relation of the concrete and the modified Menegotto-Pinto’s model (Gomes and Appleton, 1997) is used for the steel reinforcing bars.

A deformation control procedure is adopted for the analysis where the control parameter is the curvature $\kappa_n$ of the bottom cross-section. Knowing $\kappa_n$ and $N$, the neutral axis depth $x$ in Fig. 5(b) can be determined based on the axial equilibrium of the section, after which the internal moment $M_n$ can in turn be determined. As the column is statically determinate, $M_n$ defines the moments at all cross-sections, from which the curvatures of these sections and hence the deflection of the column can then be found. This deformation control procedure has many advantages as it can: follow the falling branch of the force-displacement curve; allow for cyclic responses; cope with any variation in concrete confinement; allow for P-Δ effect and geometric non-linearity; allow for the formation of a plastic hinge; and in particular, allow for slip between the plate and the RC column, that is partial interaction.
3.1 EFFECT OF COMPOSITE PLATING

In order to gauge the effects of the composite partial-interaction plating, the reinforced concrete column in Fig.6 is studied.
Properties used in the numerical analysis are given below.

**Concrete:**
- compressive strength $f_{ce} = 40$ MPa,
- tensile strength $f_{ct} = 6$ MPa,
- strain at peak strength $\varepsilon_{ce} = 0.002$,
- ultimate strain (when compressive strength=0) $\varepsilon_{ceu} = 0.006$;

**Steel plate:**
- yield strength $f_{y} = 250$ MPa,
- modulus of elasticity $E_p = 200$ GPa,
- strain hardening stiffness $E_{ph} = 600$MPa;

**Main reinforcement bars:**
- yield strength $f_{y} = 547$ MPa,
- modulus of elasticity $E_s = 200$ GPa,
- strain hardening stiffness $E_h = 600$MPa,
- coefficients used in Menegotto-Pinto’s model (Gomes and Appleton, 1997): $R_0 = 20.0$, $\alpha_1 = 19.0$, $\alpha_2 = 0.3$;

**Stirrups:**
- yield strength $f_{y} = 690$ MPa (cold pulled mild steel);

**Bolts:**
- yield strength in shear $F_{by} = 35$kN,
- elastic stiffness $K_s = 23$ kN/mm,
- strain hardening stiffness $K_{sh} = 0.7$kN/mm.

The confinement to the concrete core due to the stirrups and the P-$\Delta$ effect are not considered in the present analyses (results from Fig.7 to Fig.10) in order to isolate the effect of the composite action. These effects, however, will be incorporated later.

The effects of plating on columns are illustrated in Fig.7. The response curve marked ‘original unplated column’ is for the RC column without plating, which is used as a benchmark for direct comparison. The curve indicated ‘plate bolted 1 side’ is for the plated column shown in Fig.6, which is equivalent to that in Fig.1(a) where the plate on the right hand side has zero tension stiffness as illustrated in Fig.1(c). The case of both the tension and compression sides plated with the same plating is also studied and gives almost identical results to the ‘plate bolted 1 side’ case. The result for ‘plates glued 2 sides’ is for the case where both sides are bonded with full
interaction plates (no slip between concrete and plate) without the uplift facility illustrated in Fig.1(a), which is equivalent to adding extra fully anchored reinforcing bars. The case ‘plate bolted 1 side plus wrapping’ is for the ‘plate bolted 1 side’ column with lateral confinement from FRP wrapping. The wrapping is assumed to be able to achieve a confined concrete strength of $f_{co} = 48$ MPa (Mirmiran et al. 1998).

The mark ‘Δ’ in Fig.7 shows the stage when the concrete of the tension face first cracked; ‘×’ indicates the point where the concrete strength $f_{co}$ at strain $\varepsilon_{co}$ is just reached at the compression face; ‘+’ marks the stage of the onset of yielding of the tension reinforcing bar; ‘x’ that of yielding of the compression reinforcement bar; ‘0’ signifies the crushing of the concrete at the strain of $\varepsilon_{cu}$ or zero stress at the compression face, and ‘-’-crushing of the concrete at the position of the compression reinforcing bars. All these critical points or stages refer to the cross-section at the bottom of the column where the applied moment is greatest.

![Fig.7 Lateral responses of columns](image)

It can be seen in Fig. 7 that adding extra reinforcement, that is the case of ‘plate glued 2 sides’, can substantially increase the strength but at considerable loss of ductility which may not be beneficial in seismic retrofitting. In contrast, the system represented by the ‘plate bolted 1 side’ case in Fig.7 substantially increases the ductility with a relatively small increase in strength as is often required in seismic retrofitting in order not to increase loads on the foundations. It is also worth noting that combining wrapping with partial-interaction plating, as in the case of ‘plate bolted 1 side plus wrapping’, further increases the ductility.

Figure 8 shows the variation of axial force on the concrete alone at the bottom cross-section, that is excluding the axial forces on the reinforcement bars and the plate. It can be seen that the axial force on the concrete is reduced for the ‘plate bolted 1 side’ case compared to the ‘unplated column’, resulting in the increased deformation capacity. For the case of the ‘plate glued 2 sides’, the plates reduce the axial force on the concrete when the lateral displacement is small. However, the axial force on the concrete increases rapidly with curvature because of the increased tensile force in the tension plate. This additional axial force causes the concrete to crush earlier, as indicated by the mark ‘0’ in Figs.7&8.
Fig. 8 Variation of axial force on concrete

To further demonstrate the effect of applied axial force on the deformation capacity of the column, the column with the plate bolted to the compression face is studied in Fig. 9 with different axial loads $N$, in which $N_c = f_{cc} A_x$ and $A_x$ is the gross cross-sectional area of the RC section. The responses of the plated columns are shown with dark lines. For comparison, the original columns without plates with the same axial loads are also shown with light lines. It can be seen that partial-interaction plating increases the ductility in all cases. In addition, the effect of axial load on the ultimate deformation capacity of the columns, which may be reflected by the point ‘0’ when the concrete crushes on the compression face, is also clearly shown.

Fig. 9. Response of columns with different axial loads

Figure 10 gives the curvature distributions along the length of the column at various critical points or stages for the case of ‘plate bolted 1 side’. It can be seen that the curvature increase is concentrated in the plastic hinge from yielding of tensile reinforcement, indicated by ‘+’, to crushing of the concrete, indicated by ‘0’, whilst the curvature above the plastic hinge remains relatively unchanged. That is to say the plastic deformation of the column can be considered to only occur in the plastic hinge, something which is widely accepted in the literature. This concept will be used in the development of a displacement based design procedure in Section 5.
3.2 FACTORS AFFECTING RESPONSE

The advantages of partial interaction plating are clearly shown in the above examples. More numerical results for different plating designs are presented in this section to further study the plating effects. The column details are almost identical to those of the previous section with only slight changes to match the experimentally tested columns. (The test results are reported separately): the column length is $L=1218$mm; the first bolt is 200mm away from the bottom of the column; the spacing of the other bolts are 100mm c/c with a total number of 16 bolts at 8 cross-sections (2 bolts at a cross-section, only plated on the compression face). Confinement due to the stirrups (R6@100) is considered. The confinement effect is calculated based on the method reported by Wu, Oehlers and Griffith (2001), which gives the confined concrete strength of $f_{c} = 47$MPa that applies to the concrete core enclosed by the centre line of stirrups. The $P$-$\Delta$ effect is also included in these calculations.

Figure 11 shows the response of plated columns with different plate thicknesses but with a constant bolt stiffness ($K_b=23$kN/mm). The top curve, indicated by ‘30mm plated column with infinite stiffness’, is the result for a column plated with a 30mm thick plate having an elastic modulus and yield strength increased by 1000 times the actual values. With these values, the plate is essentially a rigid plate. Fig.12 gives the results for 6mm plated columns with a range of bolt stiffnesses.

Fig. 10 Distribution of curvature

Fig. 11 Responses of columns with different plate thickness
Fig. 12 Responses of column with different bolt stiffness (t=6mm, f_y=250MPa, K_b=0, 0.75, 5.75, 11.5, 23, 46 (kN/mm) from bottom to top)

Four more development stages are shown in Fig. 11. The point marked with ‘●’ indicates the onset of yielding of one bolt. Point ‘●’ marks the onset of yielding of all bolts. Point ‘▲’ signifies full yielding of the whole plate cross-section (for the bottom cross-section). Point ‘◆’ indicates the point where the reinforcement bar at the compression face yields in tension (the concrete compression zone is smaller than the concrete cover). The definitions of other marks are same as that in Section 3.1.

Some stages are not shown on every curve in Fig. 11, because they do not occur on every curve. For example, the bolts in the 3mm and 6mm plated columns do not yield because the plate yields first. Similarly, the plate does not fully yield for the cases of 12mm and thicker plated columns because yielding of all bolts occurs first.

For simplicity, only points ‘+’ ‘▲’ and ‘◆’ are given in Fig. 12. The plate does not yield for the case of $K_b=0.75$kN/mm in Fig. 12, as the yielding of all bolts occurs first.

The following observations can be made from Figs. 11 & 12.

1. The lateral stiffness of a column is increased invariably with an increase in plate thickness and/or bolt stiffness in the ascending branches before yielding of the tensile reinforcement. The yield points (indicated by ‘+’), which are very close the peak lateral resistance of the columns, also increase invariably with the increase of plate thickness and/or bolt stiffness.

2. There is a turning point in the lateral stiffness for all the cases when the concrete cracks in the tension zone, as indicated by the point ‘▲’ in the curves.

3. The attainment of the compressive strength at the compressive face, which is indicated by the point ‘×’, is delayed relative to the yielding of the tension reinforcement (point ‘+’) as the plate thickness increases. For example, point ‘×’ occurs before point ‘+’ for curves with $t=0$ and 3mm, whilst it occurs after point ‘+’ for cases with $t=6$mm and above. This phenomena verifies that the additional compressive resistance of the column due to plating increases when plate thickness increases. Consistently, the increase of plate thickness also delays the onset of concrete crushing at the compression face as indicated by the point “O” in Fig. 11.

4. The plating reduces the steepness of the descending slope, with thicker plates giving less steep slope, as shown in Fig. 11. Similarly, an increase of bolt stiffness also reduces the slope of the descending branch up to the point ‘▲’ where the
whole plate section yields, as shown in Fig.12. However, Fig.12 also shows that the bolt stiffness has no effect on the descending slope once the yielding of the whole plate occurs. This is reasonable since the bolt stiffness cannot further generate increases in compressive resistance from the plating system once the plate fully yields. Generally, in the range of response after yielding of the tension reinforcement and before yielding of the plating system (full yielding of either plate or bolts), the speed of degradation (reduction in lateral strength) of a column reduces when the plate thickness and/or the bolt stiffness increases.

5. The plating system improves the integrity of the column. As seen from Fig.11, the yielding of the compression reinforcement, as marked with ‘\(^*\)' and the crushing of concrete in the vicinity of compression bar, as indicated by the point ‘\(\rightarrow\)' do not occur up to the end of the chart for the 6mm and thicker plated columns. This signifies better integrity in the compression zone when compared to the original (benchmark) un-plated column. The points of ‘\(\leftarrow\)' and ‘\(^*\)' for the 3mm plated column also occur much later than that for the benchmark column.

6. The plating system improves the ultimate ductility of the column. The ultimate displacement ductility factor is defined as

\[
\mu = \frac{\Delta_u}{\Delta_y}
\]

(3.1)

where \(\Delta_y\) is the yield displacement (the point where the tensile reinforcement first yields), as indicated by the point ‘\(\rightarrow\)' and \(\Delta_u\) is the ultimate lateral displacement at the point where the lateral resistance force equals 80% of the lateral force at point ‘\(\rightarrow\)’. The ultimate ductility factors for the curves in Figs.11 & 12 are calculated and shown in Table 1.

From Table 1 it can be seen that the plating generally improves the ductility. However, increasing plate thickness or bolt stiffness does not always increase the ductility of the column. Generally, a response curve has a larger ductility factor when it has a smaller yield strength as well as a larger plateau, or more accurately, a less steep descending branch. Detailed discussions are given in Sections 3.2.1-3.2.3.

### Table 1 Ductility factors of plated columns

<table>
<thead>
<tr>
<th>Fig.11 (K_w=23) (N/m)</th>
<th>(t) (mm)</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>12</th>
<th>30</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_{py}) (MPa)</td>
<td>250</td>
<td>250</td>
<td>250</td>
<td>250</td>
<td>250</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\mu)</td>
<td>1.9</td>
<td>3.1</td>
<td>4.2</td>
<td>6.9</td>
<td>8.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Rigid plate</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Fig.12 \(K_b=6\) mm   | \(K_b\) (kN/mm) | 0.75| 5.75| 11.5| 23.0|22.0|\(\infty\)|\(\infty\) |
|------------------------|-----------------|-----|-----|-----|-----|----|-------|
| \(f_{py}\) (MPa)       | 250             | 250 | 250 | 250 | 250 | 250| 250000 |
| \(\mu\)                | 2.1             | 3.6 | 4.5 | 4.2 | 3.8 | 3.7| 8.1    |

### 3.2.1 P-Δ Effect

The steepness of the descending branch is an important factor affecting ductility. The descending slope is largely decided by the P-Δ effect. Without the P-Δ effect, the response curve of a column has a smaller descending steepness, as shown in Fig.13.
Fig. 13 P-Δ effect

The P-Δ effect causes a steepening of the descending slope by ‘rotating’ the original curve without the P-Δ effect an angle $\theta$ that satisfies $\tan(\theta) = N / L$, as shown in Fig.13, where $N$ is the axial load and $L$ is the cantilever length. This rotation of response curve due to the P-Δ effect is explained and derived by Wu, Oehler and Griffith (2001).

Therefore, columns with a larger axial load $N$ have a larger rotation angle $\theta$, and hence, a steeper descending slope. So do columns with a shorter length $L$. The later relation, with regard to the length of the column, may seem illogical according to the engineering common sense that P-Δ effect is more prominent for a longer member. The shorter columns do have a smaller lateral displacement response than that in longer columns. However, the value of a response is different from the slope of the response curve. For any given additional lateral displacement beyond the peak of the curve, the shorter columns have a larger decrease in lateral strength than the longer columns, leading to a steeper descending slope as shown in Fig.14.

Fig. 14. Effect of column length to the descending slope
3.2.2 Strength Stiffening

As seen in Fig.13, the descending branch from the point ‘+’ to the point ‘△’ is less steep than the curve after the point ‘△’. This less steep part of the curve, which extends from yielding of the tension reinforcement (point ‘+’) to yielding of the plating system, as defined by either full yielding of the plate (point ‘△’) or full yielding of the bolts (point ‘•’), is named “strength stiffened part” in this work. It is this part of curve that produces the most important advantage of plating.

From statics, the lateral force at the top of the column can be calculated by

\[ F = \frac{(M_n - N \cdot \Delta_0)}{L} \]  

(3.2)

where \( M_n \) is the moment at the bottom cross-section; \( \Delta_0 \) is the lateral displacement at the top. In the ascending part of Fig.13 before the yielding point ‘+’, \( M_n \) keeps increasing, leading to the monotonic increase of \( F \). After yielding of the column, if \( M_n \) keeps constant or reduces, \( F \) given by Eq.3.2 will decrease as and because the column displacement \( \Delta_0 \) further increases. Therefore, to keep \( F \) constant or reduce the rate of decreasing after yielding point ‘+’, \( M_n \) must increase all the time. This increase in \( M_n \) must also be big enough to counter balance the increase in the second term \( N \cdot \Delta_0 \) of Eq.3.2 in order to maintain \( F \).

For an unplated RC column, the increase in moment resistance due to the strain hardening of the tension reinforcing bars is very limited. Therefore, adequate increase in \( M_n \) is not possible unless the axial load is very small, in which case the required increase in \( M_n \) to balance \( N \cdot \Delta_0 \) is also small. However, it is possible to gain an adequate increase in \( M_n \) for a plated column even with a large axial load, as illustrated by Fig.15.

Based on the force diagram shown in Fig.2, the resisting moment of the cross-section is given by

\[ M = e_c \cdot N_{cr} + e_i \cdot N_{st} \]  

(3.3)

where \( e_c \) and \( e_i \) are eccentricities of \( N_{cr} \) and \( N_{st} \), respectively, with respect to the centroid of the cross-section and

\[ N_{cr} = N_{com} + N_{sc} + N_{pl} = N + N_{st} \]  

(3.4)

![Fig.15 Moment at the bottom section](image-url)
When the tensile reinforcement yields, $N_a$ can be considered as constant, hence $N_c$ is also a constant. Therefore, the only variable that changes in Eq.3.3 is $e_c$ and any increase in the resisting moment $M$ can only be due to an increase in $e_c$. This increase in the eccentricity $e_c$ of the compressive resultant is due to the transfer of axial force from the RC column to the plate as shown in Fig.16.

![Graph showing the distribution of axial forces in the bottom cross-section of a 6mm plated column with $K_b=23kN/mm$](image)

**Fig.16 Distribution in axial forces in bottom cross-section (6mm plated column with $K_b=23kN/mm$)**

In reality, strain hardening of the tensile reinforcement increases $N_a$ slightly, which also has an effect in increasing the moment resistance. However, the strain hardening contribution of the tension bars is small compared to the effect of the lever arm increase.

The descending slope of the strength stiffened part, as illustrated in Fig.13, is closely related to the stiffness of the plating system, i.e. the stiffness of the plate and bolt. Increasing the stiffness of the plating system reduces the descending steepness of the strength stiffened part, which can be seen from Figs.11 and 12. This is because the increase in the resisting moment $M_a$ of the cross-section due to the strength stiffening, i.e. the transfer of axial load from the RC column to the plate, is faster for stiffer plating systems. If $M_a$ increases as fast as $N \cdot \Delta_o$ does, then $M_a - N \cdot \Delta_o$ and hence $F$ keeps constant, leading to a zero descending slope. Therefore, in order to get a small descending steepness, the plating system must be able to increase the moment resistance of the bottom cross-section at a similar speed as the P-Δ effect increases the additional moment.

Once full yielding of either the plate or all bolts occurs, defined by the points '△' and '●' respectively in Figs.11 and 12, no further extension of the strength stiffening region can take place. Therefore, increasing the strength of the plate or bolts, by using thicker plates, stronger bolts or larger numbers of bolts, can extend the strength stiffening part. For example, the plate yielding point '△' occurs much later for a 6mm plated column compared to the 3mm plated column in Fig.11. Plate yielding does not occur at all for the 12mm or thicker plated columns in Fig.11, resulting in the yielding of all bolts at point '●'. However, the strength stiffening part cannot be extended indefinitely by using a stronger plating system. The reason is given below.
For a plating system that is sufficiently strong so that the plate and bolts will not fully yield under any large displacement, the strength stiffening starts, as before, with the yielding of the tensile reinforcement. However, the continuous increase in curvature at the bottom section will move the neutral axis very close towards the side of the plate if the plating system is sufficiently strong. When the compression zone is so small that the neutral axis moves to the region between the compression reinforcement and the plate, the longitudinal reinforcement at the compression side will actually be loaded in tension. In this case, $N_{sc}$ becomes a negative value and $N_{conc}$ becomes very small. This further increases the load $N_{pli}$ on the plate. In this case, $M_{3.3}$ can be re-written as

$$M = e_{conc} \cdot N_{conc} + e_{p} \cdot N_{pli} + e_{r} \cdot (N_{st} + N_{sc})$$

(3.5)

where $e_{conc}$ and $e_{p}$ are the eccentricities of $N_{conc}$ and $N_{pli}$, respectively, with respect to the centroid of the cross-section. If the compression bar yields in tension and the strain hardening is ignored, the moment terms due to $N_{sc}$ and $N_{st}$ cancel each other, as the reinforcement is assumed to be symmetrical about the centroid, and Eq.3.5 becomes

$$M = e_{conc} \cdot N_{conc} + e_{p} \cdot N_{pli}$$

(3.6)

If the axial force in the concrete is ignored as the compression zone is very small, substituting $N_{pli} = N + N_{st} - N_{sc} \approx N + (A_{st} + A_{sc}) \cdot f_{st}$ and $e_{p} \approx (D + t)/2$ into Eq.3.6 gives

$$M \approx e_{p} \cdot N_{pli} \approx \frac{D + t}{2} \left[ N + f_{st} \cdot (A_{sc} + A_{st}) \right]$$

(3.7)

At this time, the maximum moment resistance of the cross-section has been achieved. No further increase in the resistant moment, or strength stiffening, can be made regardless of how strong the plating system is. This case gives the upper limit for strength stiffening.

An example of this case is shown by the top curve in Fig.12 where the resistant moment of the bottom cross-section from the computer simulation is 82.7 kNm at the point ‘●’ where the compression bar just yields in tension, which is very close to the result given by Eq.3.7 that gives 82.4 kNm.

When the neutral axis moves to the compression concrete cover area, the whole applied axial load $N$ plus the tensile forces in all the reinforcement, which is sometimes much bigger than the axial load $N$ itself, acts on a very small compression zone. This highly stressed narrow compression zone raises concern over the stability of this region. However, experimental work has shown that no excessive distress occurs in the test columns in the compression zone and the integrity of the column remains satisfactory even after all longitudinal reinforcement yields in tension.

### 3.2.3 Yield Strength

From Table 1, it can be seen that an increase in the plating stiffness, i.e. stiffness of the bolt and the plate, does not always increase the ductility of the column. The reason is that a stiffer plating system will have a stiffer ascending response branch and greater yield strength, i.e. a greater “y” co-ordinate value at point ‘●’. The greater yield strength of the column causes a reduction of the ductility factor as calculated at a point corresponding to 80% of the yield strength in the descending branch. A stiffer plating system may also result in an earlier yielding point of the plating system, i.e. a
smaller "x" co-ordinate value at point '●' or '●', as shown in Figs.11&12. From these points of view, a plating system, that mobilises the compressive resistance only after yielding of the tensile reinforcement occurs, produces a bigger ductility factor. To verify this, a plating system with gaps between the bolts and the plate, as shown in Fig.17, is analysed. The numerical results for the 6mm plated columns with gaps of varying sizes between the plate and the bolts are given in Fig.18.

![Diagram](image_url)

**Fig.17 Gap between bolt and plate**

![Graph](image_url)

**Fig.18 Effect of gap for the 6mm plated column, $K_b=23$kN/mm**

From a ductility point of view, the advantage of the plating system with gaps is obvious from the above numerical results. The ductility factors for the cases with 1mm and 2mm gap are 4.7 and 5.0 respectively, as compared to 4.2 for the case without gap. It can be seen from Fig.18 that the initial responses of the plated columns with gaps are "exactly" the same as the un-plated one before the gap closes up at the bifurcation point where the response of the plated column splits from the response curve of the un-plated column.

Theoretically, the flexural stiffness of the plated column is different from the un-plated column due to the additional flexural stiffness of the plate that is assumed to have the same curvature as the RC column at all times. However, the flexural stiffness of the plate is negligible compared to the RC column. This explains why the plating system has no effect to the RC column before the gap closes up.

Clearly, the bifurcation point can be chosen anywhere by properly designing the width of the gap. A good gap would close up when the column just yields, as
shown by the case with 1.5mm gap in Fig.18. The slope of the response curve after
the bifurcation point can also be designed by properly choosing the rigidity of the
plate and/or the stiffness of the bolts as discussed in the above section, which is
demonstrated by the cases with $K_b=23$ & 92kN/mm in Fig.19.

Fig.19 Effect of stiffness of plating system

It has been shown that “strength stiffening” terminates when the plating
system yields, i.e. yielding of either the whole plate section or all bolts. Increasing the
yield strength of the plating system can increase the ductility of the system. As
mentioned earlier in this section, increasing the stiffness of the plating system may not
increase the ductility. It has also been shown in Section 3.2.2 and Fig.19 that a smaller
descending steepness in the strength stiffened part can be achieved by a stiffer plating
system. Therefore, a proper balance between the strength and stiffness of the plating
system is important. An idealised plating system shall be: weak or not effective before
column yielding point ‘+’; adequately stiff after yielding point to provide a small
descending steepness; and sufficiently strong to delay the yielding of the plating
system in order to achieve the maximum extent of the strength stiffened part.

A good example is demonstrated with the 6mm plated column that combines a
gap with infinite yield strength but normal stiffness for both the plate and bolts, as
shown in Fig.20. The example further demonstrates that higher yield strength for the
plate and bolts can increase the ductility of plated columns. This result suggests that
FRP materials for the plate and bolts may have advantages over steel. The strength
stiffening in Fig.20 stops at tensile yielding of the compression reinforcement shown
by the point ‘•’.

Theoretically, the strength stiffening can further be extended if the tensile
resistance in the tension face can be further increased so that the resistant moment of
the bottom cross-section further increases. Tension plating on the tension face, as
shown in Fig.21, with the tension gap closing just before point ‘•’ can be used to
serve this purpose. However, this kind of system may not be practical in the sense that
the compressive force may be too high in the compression plate. On the other hand,
the tensile strain of the reinforcement bar may also be too high in the tension side at
that stage, causing fracture of the reinforcement bar. The tensile strains at the end of
the chart for the top curve in Fig.20 are 0.0816 and 0.00387 for reinforcement bars at
the tension and the compression side, respectively.
3.2.4 Conclusions

From the studies in this section, the following conclusions on the ductility of plated columns can be drawn.

1. The ductility factor of the column is largely affected by the axial load and the length of the column. Columns with larger axial loads and/or smaller lengths will have smaller ductility factors.

2. Partial interaction plating increases the ductility factor due to the “strength stiffening” effect caused by transfer of axial load from the RC column to the plate.

3. A good plating system is less stiff, or without any stiffness, before yielding of the column, but the stiffness of the system (plate and bolts) shall be sufficient to get adequate strength stiffening after yielding of the column in order to minimise the steepness of the descending slope of load-displacement curve.

4. Yielding of the plating system (plate and bolt) stops further strength stiffening and causes the load-displacement curve to descend faster. To extend the strength stiffening part, the yielding of the plating system should be deferred as late as possible by using a stronger plating system. However, the maximum extent of strength stiffening is achieved when the reinforcement at the compression face yields in tension, after which no further strength stiffening can be obtained by increasing the strength of the plating system.
3.3 SLIP DISTRIBUTIONS

Longitudinal slip between the RC column and the plate is very important in composite structures as it reflects the degree of composite interaction between elements that are connected by bolts/shear connectors, which in turn affects the overall stress distributions in the members. Furthermore, mechanical shear connectors have only a limited slip capacity. Excessive slip will cause the fracture of the bolts/connectors. Therefore, slip and its distribution are very important considerations in the study of composite structures (Johnson and Molenstra 1991; Burnet and Oehler 2001).

Figure 22 gives the slip distributions along the length for the ‘plate bolted 1 side’ column studied in Section 3.1 at the three different stages of: concrete cracking at the stress of $f_{ct}$, as shown by the mark ‘Δ’; yielding of the tension reinforcement shown by ‘+’; and crushing of the concrete at the compressive face indicated by ‘◊’. The slip distributions at the first and second stages shown by ‘Δ’ and ‘+’ are similar to the slip distribution given by the classic linear theory of composite beams which gives a zero slip at the maximum moment position, a maximum slip at the zero moment position and a convex shaped distribution. The third distribution indicated by ‘◊’ looks quite different from the classic distribution. The figure shows that the slip near the bottom part, which is around the plastic hinge region, increases faster than that of the other part of the column. It is even possible for the slip of the first bolt to become larger than that of the remaining bolts. Fig.23 gives an example for the case where the spacing of bolts is half that for the case represented in Fig.22 and $K_{s}$ is increased from 23kN/mm to 30.5 kN/mm.

![Fig.22 Distributions of slip](image)

![Fig.23 Slip distributions when maximum slip occurs at the bottom](image)
It will be shown in Section 5.2.1 that the slip of the first bolt at the top of the plastic hinge is only determined by the deformation of the plastic hinge. The slips of the other bolts are largely determined by the stiffness and total number of bolts, i.e. the total longitudinal shear resistance of the bolts, which will be seen from the linear elastic study in Sections 4.3-4.4. Therefore, for columns with strong shear connections on the interface, i.e. stiff and/or large number of bolts, and when large deformation occurs in the plastic hinge, it is possible that the largest slip occurs at the bottom where the maximum moment occurs rather than at the top of the column where the moment is zero. This conclusion has practical implications for the design of composite members. In the literature, the classical slip distribution developed from linear elastic theory, by which the maximum slip is considered to occur at the position where the bending moment is zero, is generally adopted to guide the design of shear connectors (Oehlerls and Bradford, 1995). From the above study, it can be seen that this conclusion may not be applicable for a composite beam/column loaded to the plastic deformation stage. Further studies on the slip distribution will be conducted in Section 4.4.

From Figs.22 and 23 it can be seen that the slip distributions may be considered to be uniform along the length at the ultimate loading stage. This observation will be used to simplify the plating design procedure presented in Section 5.3.

4 MATHEMATICAL STUDIES

The advantage of partial interaction plating for RC columns has been clearly shown in the numerical studies in Section 3. In this section, further studies by mathematical analysis are performed. These analytical studies help to better understand the fundamental behaviour of plated columns. Some results from these analytical studies will also be directly used in Section 5 where the design of plated columns is discussed.

The mathematical studies of this section are based on linear elastic theory, which is applicable to serviceability limit state analysis. In order to extend the linear theory from serviceability analysis to ultimate limit state analysis, where large inelastic deformation occurs, an elastic analysis plus plastic hinge model is introduced in Section 4.1. In section 4.2, the linear elastic theory of composite columns is developed which is extended from the classic linear elastic theory of composite beams. It is shown in Section 4.3 that the response of composite members is governed by just a few composite parameters regardless of the large number of variables involved, which is a new concept firstly introduced in this work. The slip distributions are studied further in Section 4.4.

4.1 LINEAR ELASTIC PLUS PLASTIC HINGE MODEL

Linear elastic theory is only applicable when the load and deflection is relatively small. Strictly speaking, it cannot be applied when part of the structure has yielded. However, when plastic deformation occurs, most of the plastic deformation of the member is concentrated in a zone where maximum moment occurs, namely the plastic hinge zone. The plastic hinge can be considered to have a certain length for a reinforced concrete column according to the extensive experimental studies by Priestley and Park (1987) and Paulay and Priestley (1992). Therefore, the region of the column above the plastic hinge zone, as shown in Fig.24, may still be considered to be essentially elastic where linear-elastic theory is applicable.
For the elastic member above the plastic hinge, the slip at the imaginary support or the interface of the elastic and the plastic part, as shown in Fig.24(b), will not be zero. Therefore a non-zero slip boundary condition, as will be discussed in Section 4.2.2, must be used in the linear theory. This boundary slip $s_p$ can be calculated by the plastic hinge analysis introduced in Section 5.2. Therefore, by using the model shown in Fig.24, the difficult problem of a partial interaction non-linear analysis is transformed into a relatively simple linear-elastic analysis of the member with a shorter length $L_e$ and a given slip $s_p$ at the support of the elastic member, plus a plastic hinge analysis.

![Fig.24 Plastic-elastic model](image)

**4.2 LINEAR ELASTIC ANALYSIS**

Classic linear elastic partial-interaction theory for composite steel and concrete beams was first developed by Newmark et al (1951, 1952) and more recently extended to allow for non-linearity by Johnson and Molenstra (1991) and Burnet and Oehler (2001). In this study, Newmark's classic linear-elastic theory for composite beams is extended to include axial load so that it can be applied to plated columns as well as prestressed RC composite beams. Furthermore, a non-zero slip boundary condition at the fixed end support is introduced to allow Newmark's linear elastic serviceability analysis to be extended to encompass beams and columns at the ultimate limit state where large deformations occur at the plastic hinge, using the model of Section 4.1.

**4.2.1 Generic Mathematical Model**

The basic geometric model under consideration is shown in Fig.25. The origin of the co-ordinate system is located at the geometric centroid of the cross-section of element 1 at the top of the column. The $x$ axis is in the longitudinal direction of the column. A
typical portion of the column with length \(x\) from the top is isolated as a free body as shown in Fig.25(a).

![Free body diagram and strain profile](image)

(a) Free body diagram         (b) Strain profile in the section

Fig.25 Analytical model

A constant axial load \(N\) is applied at the centroid of the cross-section of element 1. For generality, a constant moment \(M_0\) at the origin and a distributed load \(p(x)\) along the length are applied. In the free body diagram of Fig.25(a), the normal (\(y\) direction) stress on the interface is not shown as it has no effect in the following derivations.

The theory developed in this section is based on the following assumptions:

1. All the constitutive materials behave linearly;
2. The cross-section is uniform along the length;
3. Bernoulli’s principle that plane sections remain plane applies to individual elements, as shown by the strain profile in Fig.25(b);
4. The shear connectors between the two elements are continuous and uniformly distributed longitudinally; and
5. No transverse separation occurs on the contact interface, therefore the curvature is the same for both elements at the same point.

### 4.2.1.1 Equilibrium and compatibility

Force equilibrium for the individual elements in Fig.25 gives

\[
N_2 - F_{\text{str}} = 0 \tag{4.1}
\]

\[
N - F_{\text{str}} - N_1 = 0 \tag{4.2}
\]

Taking moments about point A gives

\[
M_1 + M_2 + N_2 \cdot (h_2 + h_1) - M(x) = 0 \tag{4.3}
\]

where \(M(x)\) is applied the moment in the cross-section given by

\[
M(x) = M_0 + F \cdot x - \int p(\xi) \cdot (x - \xi) d\xi \ . \text{Hence from Eq.4.1 and Eq.4.3}
\]

\[
M_1 + M_2 + F_{\text{str}} \cdot (h_2 + h_1) - M(x) = 0 \tag{4.4}
\]

The strain in the x-direction at the interface of element 1 in Fig.25(b) is given by

\[
\varepsilon_i = \frac{M_1 \cdot h_i}{(EI)_i} + \frac{N_1}{(EA)_i} \tag{4.5}
\]
and that at the interface of element 2 is

\[ \varepsilon_s = -\frac{M_2 \cdot h_2 + N_2}{(EI)_2} \cdot \frac{1}{(EA)_2} \]  \hspace{1cm} (4.6)

Assumption 5 requires that

\[ \kappa = \frac{M_1}{(EI)_1} = \frac{M_2}{(EI)_2} \]  \hspace{1cm} (4.7)

Applying Assumption 4, the longitudinal slip \( s \) is given by

\[ s = \frac{F_b}{K_b} = \frac{q \cdot L_s}{K_b} \]  \hspace{1cm} (4.8)

where \( F_b \) is the shear force applied to the shear connectors at a given cross-section, which has a stiffness of \( K_b \); \( L_s \) is the longitudinal spacing of the shear connectors; and \( q \) is the longitudinal shear force per unit length, or shear flow, which is given by

\[ q = \frac{dF_{shr}}{dx} \]  \hspace{1cm} (4.9)

Hence

\[ s = \frac{L_s}{K_b} \cdot \frac{dF_{shr}}{dx} \]  \hspace{1cm} (4.10)

The slip strain in Fig. 25(b) is given by

\[ \varepsilon_{slp} = \varepsilon_1 - \varepsilon_2 = -\frac{ds}{dx} \]  \hspace{1cm} (4.11)

Differentiating Eq. 4.10 gives

\[ \frac{ds}{dx} = \frac{L_s}{K_b} \cdot \frac{d^2F_{shr}}{dx^2} \]  \hspace{1cm} (4.12)

By referring to Eqs. 4.1, 4.2, 4.4-4.7, the slip strain in Eq. 4.11 can be transformed to

\[ \varepsilon_{slp} = \varepsilon_1 - \varepsilon_2 = -\frac{EI}{EA \cdot \sum EI} \cdot F_{shr} + \frac{(h_1 + h_2) \cdot M(x)}{\sum EI} + \frac{N}{(EA)_1} \]  \hspace{1cm} (4.13)

in which

\[ \frac{1}{EI} = \frac{1}{(EA)_1} + \frac{1}{(EA)_2} \]  \hspace{1cm} (4.14)

\[ \sum EI = (EI)_1 + (EI)_2 \]  \hspace{1cm} (4.15)

and

\[ \overline{EI} = \sum EI + \overline{EA} \cdot (h_1 + h_2)^2 \]  \hspace{1cm} (4.16)

where \( \overline{EI} \) is the flexural rigidity of the composite member with full interaction. However, \( \overline{EA} \) is not the axial rigidity of the composite member.

### 4.2.1.2 Governing differential equation

From Eqs. 4.11, 4.12 and 4.13, the following differential equation is obtained

\[ \frac{L_s}{K_b} \cdot \frac{d^2 F_{shr}}{dx^2} - \frac{\overline{EI}}{EA \cdot \sum EI} \cdot F_{shr} + \frac{(h_1 + h_2) \cdot M(x)}{\sum EI} + \frac{N}{(EA)_1} = 0 \]  \hspace{1cm} (4.17)

Letting

\[ a_b = \frac{L_s}{K_b} \]  \hspace{1cm} (4.18)
\[ a_1 = \frac{EI}{EA \cdot \sum EI} \]  
\[ a_2 = \frac{(h_1 + h_2)}{\sum EI} \]  
\[ a_3 = \frac{1}{(EA)_1} \]

Eq.4.17 simplifies to
\[ a_0 \cdot \frac{d^2 F_{sh}}{dx^2} - a_1 \cdot F_{sh} + a_2 \cdot M(x) + a_3 \cdot N = 0 \]  
(4.22)

This differential equation can be solved for a given applied moment distribution of \( M(x) \). The slip then can be obtained from Eq.4.10. To obtain the deflection of the member, the following relation is used
\[ -\frac{d^2v}{dx^2} = \kappa = \frac{M_1}{(EI)_1} \]  
(4.23)

Combining Eqs.4.4, 4.7 and 4.17 to give
\[ \frac{M_1}{(EI)_1} = \frac{M(x)}{EI} - \frac{(h_1 + h_2) \cdot EA}{EI} \cdot \frac{L_s}{K_s} \cdot \frac{d^2 F_{sh}}{dx^2} - \frac{(h_1 + h_2) \cdot EA \cdot N}{EI \cdot (EA)_1} \]  
(4.24)

Substituting Eq.4.24 into Eq.4.23 gives
\[ \frac{d^2v}{dx^2} = a_4 \cdot a_0 \cdot \frac{d^2 F_{sh}}{dx^2} - \frac{M(x)}{EI} + a_4 \cdot a_3 \cdot N \]  
(4.25)

where
\[ a_4 = \frac{(h_1 + h_2) \cdot EA}{EI} \]  
(4.26)

### 4.2.2 Solution For The Case Of A Cantilever Column

![Fig.26 Model of the cantilever column](image)
The solution for the cantilever column shown in Fig. 26 is just a special case of Section 4.2.1. In this case, \( M(x) = F \cdot x \) and the general solution of Eq. 4.22 is given by

\[
F_{sbr} = c_1 \cdot e^{\sqrt{\frac{a_2 \cdot L}{a_1}}} + c_2 \cdot e^{-\sqrt{\frac{a_2 \cdot L}{a_1}}} + \frac{a_2 \cdot F}{a_1} \cdot x + \frac{a_3 \cdot N}{a_1} \quad (4.27)
\]

where the constants \( c_1 \) and \( c_2 \) can be determined from the following boundary conditions:

\[
F_{sbr} \bigg|_{x=0} = 0 \quad (4.28)
\]

\[
s_{b L} = \frac{L_1}{K_b} \cdot \frac{dF_{sbr}}{dx} \bigg|_{x=L} = s_p \quad (4.29)
\]

For a fixed end support as shown in Fig. 26, the boundary slip \( s_p = 0 \) at \( x = L \). For generality, a non-zero value \( s_p \) is adopted. This non-zero boundary slip condition is useful as discussed in Section 4.1 where the linear theory is extended to a non-linear analysis. From Eqs. 4.28 and 4.29, the constants are calculated to be

\[
c_1 = \frac{s_p}{2 \cdot \sqrt{a_0 \cdot a_1} \cdot \cosh(\alpha)} - \frac{a_3 \cdot N \cdot e^{\alpha} + a_2 \cdot F \cdot \sqrt{\frac{a_0}{a_1}}}{2a_1 \cdot \cosh(\alpha)} \quad (4.30)
\]

\[
c_2 = -c_1 - \frac{a_3 \cdot N}{a_1} \quad (4.31)
\]

in which

\[
\alpha = L \cdot \sqrt{\frac{a_1}{a_0}} = L \cdot \sqrt{a_0} = L \cdot \frac{2L_1 \cdot E I}{L_1 \cdot EA \cdot \sum EI} \quad (4.32)
\]

where

\[
a_0 = \frac{a_1}{a_0} \quad (4.33)
\]

Substituting \( c_1 \) and \( c_2 \) into Eq. 4.27 and re-arranging gives

\[
F_{sbr} = \frac{s_p \cdot \sinh(\alpha \cdot \xi)}{\sqrt{a_0 \cdot a_1} \cdot \cosh(\alpha)} + \frac{1}{\alpha} \left[ a_3 \cdot N \cdot (1 - e^{\alpha \cdot \xi}) - \frac{2 \cdot \sinh(\alpha \cdot \xi)}{e^{2\alpha} + 1} + a_2 \cdot F \cdot \sqrt{\frac{a_0}{a_1}} \cdot (\alpha \cdot \xi - \frac{\sinh(\alpha \cdot \xi)}{\cosh(\alpha)}) \right]
\]

\[
= F_f + F_n + F_{sp} \quad (4.34)
\]

where \( F_f \), \( F_n \) and \( F_{sp} \) are the longitudinal shear forces induced by the lateral force \( F \), the axial load \( N \) and the boundary slip \( s_p \), respectively. They are given by

\[
F_f = F \cdot L \cdot (h_1 + h_2) \cdot \frac{E A}{E I} \left[ \xi - \frac{\sinh(\alpha \cdot \xi)}{\alpha \cdot \cosh(\alpha)} \right] \quad (4.35)
\]

\[
F_n = \frac{N}{(EA)_1} \cdot \frac{E A \cdot \sum EI}{E I} \left[ 1 - e^{-\alpha \cdot \xi} - \frac{2 \cdot \sinh(\alpha \cdot \xi)}{e^{2\alpha} + 1} \right] \quad (4.36)
\]

\[
F_{sp} = s_p \cdot L \cdot \frac{K_b}{L_1} \cdot \frac{\sinh(\alpha \cdot \xi)}{\alpha \cdot \cosh(\alpha)} \quad (4.37)
\]

where \( \xi \) is the normalized coordinate \( x \), or
\[ \zeta = \frac{x}{L} \quad 0 \leq \zeta \leq 1 \]  
(4.38)

The slip is obtained from Eq.4.10

\[ s = \frac{L}{K_b} \cdot \frac{dF_{sp}}{dx} = s_f + s_n + s_{sp} \]  
(4.39)

where \( s_f, s_n \) and \( s_{sp} \) are the slip terms due to \( F, N \) and \( s_p \), respectively, and are given by

\[ s_f = F \cdot (h_1 + h_2) \cdot \frac{EA}{EI} \cdot \frac{L}{K_b} \left[ 1 - \frac{\cosh(\alpha \cdot \zeta)}{\cosh(\alpha)} \right] \]  
(4.40)

\[ s_n = \frac{N}{(EA)} \cdot \sqrt{\frac{L}{EI}} \cdot \sum EI \cdot \left[ e^{\alpha \cdot \zeta} - \frac{2 \cosh(\alpha \cdot \zeta)}{e^{2\alpha} + 1} \right] \]  
(4.41)

\[ s_{sp} = \frac{s_p \cdot \cosh(\alpha \cdot \zeta)}{\cosh(\alpha)} = \frac{s_p \cdot \cosh(\alpha \cdot \zeta)}{\cosh(\alpha \cdot L)} \]  
(4.42)

To solve for the deflection, Eq.4.25 is integrated twice to give

\[ v = v_{full} + a_3 \cdot a_0 \cdot F_{stv} + \frac{a_4 \cdot a_5 \cdot N}{2} \cdot x^2 + c_3 \cdot x + c_4 \]  
(4.43)

where

\[ v_{full} = \int dx \left( -\frac{M}{EI} \right) = \int dx \left( -\frac{F \cdot x}{EI} \right) \]  
(4.44)

which is the deflection due to the lateral force \( F \) with full interaction, i.e. no slip between elements 1 and 2. The constants \( c_3 \) and \( c_4 \) are determined from the boundary conditions:

\[ \frac{dv}{dx} \bigg|_{x=0} = 0 \]  
(4.45)

\[ \frac{dv}{dx} \bigg|_{x=L} = 0 \]  
(4.46)

which gives

\[ c_3 = -a_4 \cdot a_3 \cdot L \cdot N - a_4 \cdot a_0 \cdot \frac{s_p \cdot K_b}{L_s} \]  
(4.47)

\[ c_4 = \frac{a_4 \cdot a_3 \cdot L^2 \cdot N}{2} - a_4 \cdot a_0 \cdot \left[ 2c_1 \sinh(\alpha) + \frac{a_5 \cdot N}{a_1} \cdot (1 - e^{-\alpha}) + \frac{a_2 \cdot F \cdot L}{a_1} \cdot s_p \cdot \frac{L \cdot K_b}{L_s} \right] \]  
(4.48)

Substituting \( c_3 \) and \( c_4 \) into Eq.4.43 and re-arrangement gives

\[ v = v_{full} + v_{v} + F \cdot g_s(\zeta) + N \cdot g_2(\zeta) + s_p \cdot g_3(\zeta) \]  
(4.49)

where
\[

\nu_{\text{full}} = \frac{(h_1 + h_2) \cdot EA \cdot L^2}{2EI \cdot (EA)_1} \cdot N \cdot (\xi - 1)^2 \\
= \frac{1}{2} \cdot N \cdot a_4 \cdot a_2 \cdot L^2 \cdot (\xi - 1)^2 \\

\text{(4.50)}

\]

which is the deflection caused by a constant moment due to the eccentricity of the axial load about the centroid of the combined composite cross-section of elements 1 and 2 with full interaction.

The three additional terms in Eq.4.49, \( F \cdot g_1(\xi) \), \( N \cdot g_2(\xi) \) and \( s_p \cdot g_3(\xi) \), reflect the partial interaction effects for the lateral force, axial force and boundary slip, respectively, where

\[
g_1(\xi) = (h_1 + h_2)^2 \left( \frac{EA}{EI} \right)^2 \cdot \frac{L}{K_b} \cdot L \cdot \left[ \xi - \frac{\sinh(\alpha \cdot \xi)}{\alpha \cdot \cosh(\alpha)} - 1 + \frac{1}{\alpha} \cdot \tanh(\alpha) \right] \\
= a_4^2 \cdot a_6 \cdot L \cdot \left[ \xi - \frac{\sinh(\alpha \cdot \xi)}{\alpha \cdot \cosh(\alpha)} - 1 + \frac{1}{\alpha} \cdot \tanh(\alpha) \right] \\
\text{(4.51)}
\]

\[
g_2(\xi) = (h_1 + h_2) \left( \frac{EA}{EI} \right) \sum \frac{EI \cdot L}{K_b} \cdot \left[ -e^{-\alpha \xi} - \frac{2\sinh(\alpha \cdot \xi)}{e^{2\alpha} + 1} + \frac{1}{\cosh(\alpha)} \right] \\
= \frac{a_3 \cdot a_5}{a_4} \left[ -e^{-\alpha \xi} - \frac{2\sinh(\alpha \cdot \xi)}{e^{2\alpha} + 1} + \frac{1}{\cosh(\alpha)} \right] \\
\text{(4.52)}
\]

\[
g_3(\xi) = (h_1 + h_2) \left( \frac{EA}{EI} \right) \cdot L \cdot \left[ \frac{\sinh(\alpha \cdot \xi) - \sinh(\alpha)}{\alpha \cdot \cosh(\alpha)} + 1 - \xi \right] \\
= a_4 \cdot L \cdot \left[ \frac{\sinh(\alpha \cdot \xi) - \sinh(\alpha)}{\alpha \cdot \cosh(\alpha)} + 1 - \xi \right] \\
\text{(4.53)}
\]

The functions \( g_1(\xi) \) and \( g_2(\xi) \) approach zero when the stiffness of shear connectors approaches infinity, which leads to the result given by full interaction theory.

### 4.3 COMPOSITE PARAMETERS

For a composite member, there are many member properties that affect the response. Furthermore, the effect of each variable is not obvious. Some of the variables are inter-related, further complicating matters. For example, in Eq.4.40 the slip appears to be in direct proportion to \((h_1 + h_2)\), however, \(EI\) and \(EA\) are also related to \((h_1 + h_2)\). Furthermore, \(\alpha\) also depends on \(h_1\) and \(h_2\). However, careful study of the variables reveals that the response of a composite member is determined by only a few composite parameters that are combinations of basic material and geometric properties as described in the following sections.

### 4.3.1 Fundamental Parameters Governing Longitudinal Slip

In order to establish which factors affect the slip distribution in a composite member with arbitrary loading and boundary conditions, Eq.4.13 is rewritten with reference to Eqs.4.19-4.21 as

\[
\varepsilon_{slip} = -a_1 \cdot F_{slip} + a_2 \cdot M(x) + a_3 \cdot N \\
\text{(4.54)}
\]
For a linear system, the superposition law applies. Therefore, the total slip strain of Eq.4.54 can be considered to be the algebraic sum of the following 3 components

\[
(\varepsilon_{slp})_1 = -\frac{ds_1}{dx} = -a_1 \cdot F_{shr} \tag{4.55}
\]

\[
(\varepsilon_{slp})_2 = -\frac{ds_2}{dx} = a_2 \cdot M(x) \tag{4.56}
\]

\[
(\varepsilon_{slp})_3 = -\frac{ds_3}{dx} = a_3 \cdot N \tag{4.57}
\]

where \((\varepsilon_{slp})_1\), \((\varepsilon_{slp})_2\) and \((\varepsilon_{slp})_3\) are the slip strain terms caused by (1) longitudinal shear force \(F_{shr}\), (2) bending moment \(M(x)\) and (3) axial load \(N\), respectively. The corresponding slip terms are indicated by \(s_1\), \(s_2\) and \(s_3\), respectively. Therefore

\[
\varepsilon_{slp} = (\varepsilon_{slp})_1 + (\varepsilon_{slp})_2 + (\varepsilon_{slp})_3 \tag{4.58}
\]

\[
s = s_1 + s_2 + s_3 \tag{4.59}
\]

Integrating Eq.4.56 and 4.57 gives

\[
s_2 = -a_2 \cdot \int M(x) \cdot dx = a_2 \cdot F_2(x) \tag{4.60}
\]

\[
s_3 = -a_3 \cdot \int N \cdot dx = a_3 \cdot F_3(x) \tag{4.61}
\]

where \(F_2(x) = -\int M(x) \cdot dx\) and \(F_3(x) = -\int N \cdot dx\). By referring to Eq.4.10, Eq.4.55 can be rewritten as

\[
-\frac{ds_1}{dx} = -a_1 \cdot \frac{1}{a_0} \int s \cdot dx \tag{4.62}
\]

Taking derivative both sides and referring to Eqs.4.33, 4.59-4.61 gives

\[
\frac{d^2 s_1}{dx^2} - a_3 \cdot s_1 - a_3 \cdot [a_2 \cdot F_2(x) + a_3 \cdot F_3(x)] = 0 \tag{4.63}
\]

The factors affecting the slip can now be clearly seen from Eqs.4.59-4.61 and (4.63). Apart from the loading conditions which determine \(F_2(x)\) and \(F_3(x)\), there are only three composite parameters that affect the slip: \(a_2\), \(a_3\) and \(a_3\). These three composite parameters are functions of geometric and material properties of the member. Furthermore, these parameters have clear physical meanings as discussed below.

1. **The unit longitudinal flexibility of shear connectors \(a_0\).**

   As given by Eq.4.18, it can be seen that \(1/a_0\) is the shear connector lateral stiffness over a unit longitudinal length. In other words, \(a_0\) is the flexibility coefficient of shear connection. Therefore, a larger value of \(a_0\) indicates a smaller resistance to slip from the shear connectors, tending to give a larger value of slip in the interface.

2. **The active slip strain coefficient \(a_2\).**

   Letting \(F_{shr} = 0\), \(N=0\) and \(M=I\) in Eq.4.54 gives \(\varepsilon_{slp} = a_2\), where \(a_2\) is given by Eq.4.20. This means that the coefficient \(a_2\) is the slip strain caused by a unit external bending moment at the cross-section when no interaction exists at the interface (no shear connectors). A larger value of \(a_2\) means that, for a given external bending moment, a larger slip strain will occur. In other words, a member with a larger \(a_2\) value is prone to have larger slip under flexural loading. From
Eq.4.20, it can be seen that \( a_2 \) is a cross-sectional property of the individual elements 1 and 2.

3. The passive slip strain coefficient \( a_1 \).
   Let \( F = N = 0 \) and \( F_{\text{sl}} = 1 \) in Eq.4.54, then \( \varepsilon_{\text{sl}} = -a_1 \), in which \( a_1 \) is also a function of the cross-sectional properties as given by Eq.4.19. Therefore, the coefficient \( a_1 \) is the slip strain of the cross-section caused by a unit longitudinal shear force. For a cross-section with a larger \( a_1 \), a given longitudinal shear force will cause larger longitudinal slip strains and hence slip. As the longitudinal shear force always tries to stop the slip caused by the external forces (hence the name passive slip strain coefficient as opposed to the active slip strain coefficient \( a_2 \)), a larger \( a_1 \) indicates that the longitudinal shear is more effective in stopping the slip or resisting the slip strain caused by external forces. In other word, for composite members of the same length but with a larger cross-sectional property \( a_1 \), the same shear connection (same bolt and same number) will result in a smaller slip at the interface under the same loading conditions.

4. The axial flexibility \( a_3 \).
   Letting \( F_{\text{sl}} = 0 \), \( N = 1 \) and \( M = 0 \), then Eq.4.54 gives \( \varepsilon_{\text{sl}} = a_3 \), where \( a_3 \) is given by Eq.4.21. That is to say, \( a_3 \) is the slip strain caused by a unit axial force applied at the centroid of element 1. When the axial flexibility \( a_3 \) of element 1 is greater, a given axial load will cause a larger axial shortening of element 1 which in turn causes a larger slip between elements 1 and 2.

5. The slip resistance ability coefficient \( a_4 \).
   As given by Eq.4.33, this coefficient is the combination of \( a_0 \) and \( a_4 \), i.e. \( a_4 = a_4/a_0 \). It reflects the overall ability of the composite member in restraining the slip at the interface, which can be observed by considering the physical meanings of \( a_0 \) and \( a_4 \). As mentioned in 3, composite members with a larger passive slip strain coefficient \( a_4 \) and the same shear connection will have a smaller slip at the interface under the same loading conditions. When the shear connector stiffness \( 1/a_0 \) is larger, there is a stronger shear connection in the interface, providing greater slip resistance, which further reduces the slip. Therefore, for composite members with larger \( a_4 \), the ability of the member to restrain the slip is greater, and the slip caused by a given loading condition will be smaller.

The above relations can be verified by the slip results from the example of Section 4.2.2 where the slip terms are given by Eqs.4.40-4.42. In all three terms, which respectively are caused by the external moment or lateral force at the top (Eq.4.40), the axial load (Eq.4.41) and the boundary slip (Eq.4.42), the slip resistance ability coefficient \( a_4 \) plays a key role in resisting the amount of slip. In other word, all three slip terms given by Eq.4.40-4.42 reduce monotonically when \( a_4 \) increases, which will be discussed further in Sections 4.4.1-4.4.3.
4.3.2 Parameters Affecting Deformations

Similar to slip, the deflection of a composite member is also governed by just a few key parameters regardless of the many variables involved. With reference to Eqs. 4.11 and 4.12, Eq. 4.24 can be rewritten as:

$$
\kappa = \frac{M(x)}{EI} + \frac{(h_1 + h_2) \cdot EA}{EI} \cdot \varepsilon_{slp} - \frac{(h_1 + h_2) \cdot EA}{EI \cdot (EA)_1} \cdot N \\
= \frac{M(x)}{EI} + a_4 \cdot \varepsilon_{slp} - a_4 \cdot a_1 \cdot N \tag{4.64}
$$

where $a_4$ is given by Eq. 4.26. The first term in the above equation is the curvature caused by the external moment with full interaction (no slip). The second is the additional curvature of the cross-section caused by a given slip strain $\varepsilon_{slp}$. And the third term is the curvature caused by the axial load due to the eccentricity between the centroid of element 1 and the centroid of the composite section with full interaction.

From Eq. 4.64, the factors affecting the curvature, and hence deformation, of the member are clearly seen. Apart from the external member forces $N$ and $M(x)$ as well as the full interaction flexural rigidity $\bar{EI}$, the only other composite parameters that affect the deformations are $a_2$, $a_3$, $a_5$ and $a_4$, as the slip strain $\varepsilon_{slp}$ is determined by $a_2$, $a_3$, $a_5$, and loadings as discussed in Section 4.3.1. This observation is verified by the deflection results given by Eqs. 4.49-4.53 and 4.44.

Similar to $a_2$, $a_3$, and $a_5$, there is also a clear physical meaning for coefficient $a_4$. To visualize this physical meaning, let $M(x)=0$ and $N=0$ in Eq. 4.64 and consider Eq. 4.54 to get

$$
\kappa = a_4 \cdot \varepsilon_{slp} = -a_4 \cdot a_1 \cdot F_{shr} \tag{4.65}
$$

For convenience, the corresponding free body diagram and strain profile in this case is shown in Fig. 27. The strain distribution of element 1 is given by $\varepsilon = \frac{N_1}{(EA)_1} - \kappa \cdot y$.

Considering Eq. 4.65 and $N_1 = -F_{shr}$ leads to

$$
\varepsilon = \frac{F_{shr}}{(EA)_1} + a_1 \cdot a_4 \cdot F_{shr} \cdot y = F_{shr} \cdot (a_1 \cdot a_4 \cdot y - a_2) \tag{4.66}
$$

![Fig. 27 Strain profile due to longitudinal shear force](image-url)
It is noted that at the point where \( a_i \cdot a_4 \cdot y - a_3 = 0 \) in Eq.4.66, \( \varepsilon = 0 \) for any value of \( F_{shr} \). The y co-ordinate of this point is given by

\[
y_1 = \frac{a_3}{a_i \cdot a_4}
\]

(4.67)

In other words, the additional strain caused by longitudinal shear (or slip) is always zero at this point no matter how the longitudinal shear force or slip changes. This means that all the strain profiles, for any combination or distribution of shear connections, pass through this point or are "focused at this point".

The strain profile ‘focal point’ is an important new concept in the study of composite structures that was first discovered and introduced by Seracino, Oehlers and Yeo (2001). It was found initially from their numerical simulations. They later verified this finding using linear elastic analysis to show that all strain profiles calculated from the elastic theory, with different shear connections, intersected at a common point. However, the physical reason as to why the focal point exists was still not clear from their study.

From the above discussion, it can now be seen clearly that the longitudinal shear force \( F_{shr} \) causes an additional axial force \( N_i = -F_{shr} \) and an additional moment \( M_i = -a_4 \cdot a_i \cdot F_{shr} \cdot (EI)_1 \) in the cross-section of element 1. This additional force and moment lead to the additional strain of \( -F_{shr}/(EA)_1 \) and \( a_4 \cdot a_i \cdot F_{shr} \cdot y \), respectively, in the cross-section, which cancel each other at the focal point regardless of the value of \( F_{shr} \) that is affected by the shear connection details.

This investigation also reveals that the ‘focal point’ concept is, strictly speaking, the result of linear systems. The linear strain distribution given by Eq.4.66 is not applicable to a general non-linear system which cannot have a single \( y \) value for \( \varepsilon = 0 \) as was used to get Eq.4.67 in the above derivation.

For the same reason, element 2 also has a focal point. The y co-ordinate of focal point 2 can be easily calculated as

\[
y_2 = \frac{1}{a_i \cdot a_4 \cdot (EA)_2} - h_1 - h_2
\]

(4.68)

The distance between these two focal points, \( R_s \) as shown in Fig.27, is given by

\[
R_s = y_1 - y_2 = \frac{1}{a_i \cdot a_4 \cdot (EA)_1} + \frac{1}{a_i \cdot a_4 \cdot (EA)_2} + h_1 + h_2
\]

\[
= \frac{1}{a_4}
\]

(4.69)

From Eq.4.65, it can be seen that \( a_4 \) is the curvature when the slip strain caused by the longitudinal shear \( \varepsilon_{shr} = 1 \). Because of this reason, \( a_4 \) is named as the unit slip curvature. Therefore, according to Eq.4.69, \( R_s \) is named the slip radius as oppose to the unit slip curvature \( a_4 \). It is noted that the slip radius (and unit slip curvature) is also function of cross-sectional properties.

From Fig.27, it can also be seen that the slip strain can be expressed as the slip radius times the curvature caused by the longitudinal shear force, or

\[
\varepsilon_{shr} = R_s \cdot \kappa = \frac{\kappa}{a_4}
\]

(4.70)
Comparing Eq.4.70 with Eq.4.65, they are the same which further supports the concept of ‘focal points’. Interestingly, the first and third terms in Eq.4.64 can also be obtained with the concept of slip radius or unit slip curvature. For a composite member without interaction, the curvature due to an external moment is given by \( \kappa = \frac{M(x)}{\sum EI} \). The slip strain at the interface is simply

\[
\varepsilon_{slp} = \frac{M(x)}{\sum EI} \cdot (h_1 + h_2) \quad (4.71)
\]

For a composite member with full interaction, this slip strain must be resisted by the longitudinal shear force so that \( \varepsilon_{slp} = 0 \), or the slip strain due to the longitudinal shear force is

\[
\varepsilon_{slp} = -\frac{M(x)}{\sum EI} \cdot (h_1 + h_2) \quad (4.72)
\]

From Eq.4.70, the additional curvature due to longitudinal shear is given by

\[
a_4 \cdot \varepsilon_{slp} = -a_4 \cdot \frac{M(x)}{\sum EI} \cdot (h_1 + h_2) \quad (4.73)
\]

The final curvature, assuming full interaction, is therefore the summation of the two parts, or

\[
\kappa = \frac{M(x)}{\sum EI} - a_4 \cdot \frac{M(x)}{\sum EI} \cdot (h_1 + h_2) = -\frac{M(x)}{EI} \quad (4.74)
\]

which is the same as the first term in Eq.4.64. The third term of Eq.4.64 can also be obtained with the same concept.

4.4 SLIP DISTRIBUTION OF THE CANTILEVER COLUMN

Due to the importance of the longitudinal slip, its distributions are further discussed in this section based on the results from linear elastic theory. Typical slip distributions for the cantilever column obtained from Section 4.2.2, as given by Eqs.4.40-4.42, are shown in Fig.28.

![Fig.28 Typical slip distributions](image)

The slip term due to flexural deformation given by Eq.4.40 is shown by the curve marked with “by F” in Fig.28. The slip term caused by the axial load \( N \) given by
Eq.4.41 is marked with "by N" in the figure. When the load and deflection is small, no plastic hinge forms and the total slip distribution is given by the curve marked with "by F+N". The other term given by Eq.4.42 is induced by the boundary slip $s_p$ at the elastic-zone/plastic-hinge interface when plastic hinge forms at large deformations, as marked with "by $s_p$." in Fig.28. Summation of the above three terms depicts the total slip when large plastic deformation occurs, as shown by the thick line indicated with "Total" in Fig.28.

After the formation of the plastic hinge, the slip on top of the plastic hinge, $s_p$, in Fig.28, continues to increase when the deflection of the column is further increased. The reason that $s_p$ only depends on the deformation of the plastic hinge will be shown in Section 5.2.1. However, the slip term due to $N$ does not change. The slip term due to $F$ is actually determined by the moment distribution along the column. It may continue to increase slightly due to the moment increase in the cross-sections after yielding of the column, but is relatively more stable as compared to the fast increase of slip $s_p$ caused by the fast increase of plastic deformation in the hinge. It is, therefore, possible that when the plastic hinge deformation of the column is sufficiently large, which is directly related to the slip $s_p$, the slip term 'by $s_p$' is more prominent than the other two terms. In this case, the position of maximum slip will occur at the top of the plastic hinge instead at the top of the column.

An example that compares the analytical results to the numerical results, produced by the computer program "PLTCOL", is given in Fig.29.

![Fig. 29 Slip distributions compared with the numerical results](image)

The numerical slip distribution just before the concrete cracks at the tension face, as indicated by 'Δ', agrees very well with the analytical results. This means that the deformation of the column can be reasonably modelled as linear elastic before the concrete cracks on the tension face. At large deformations, however, the numerical results are generally bigger than the analytical ones, as shown by the curve marked with '○' which indicates the stage just before the concrete crushes (compressive strength and hence stress=0) on the compression face. This difference is mainly due to the assumption made in Section 4.1 that the column is still linear elastic above the plastic hinge at large deformations. The small non-linearity in the part of the column above the plastic hinge causes the numerical or actual column to deform slightly more
than predicted by elastic theory. The slight increase of column deformation corresponds to an extra amount of slip. This is the reason that the analytical slip distribution at the large deformation stage underestimates the amount of slip. However, the numerical results may slightly overestimate the true slip due to the neglecting of tension stiffening effect. If tension stiffening were considered, the column would be slightly stiffer hence experiencing less deflection, which would lead to a smaller slip. That is to say there would be a closer agreement between analytical and numerical results if tension stiffening were considered in the numerical calculations. As the difference in Fig.29 is not substantial, this comparison verifies the legitimacy of the linear assumption outside the plastic hinge region.

4.4.1 Slip Due To Flexural Moment

From the slip term caused by the flexural moment as given by Eq.4.40, the slip \( s_f \):

- decreases monotonically when the slip resistance ability coefficient \( a_s \) increases. This relation is shown in Fig.30(a) by the function \( y = \frac{1}{a_s} \left[ 1 - \cosh(L \cdot \sqrt{a_s} \cdot \xi) / \cosh(L \cdot \sqrt{a_s}) \right] \) which is part of the slip term. There are two asymptotes for the curve in Fig.30(a). Mathematically it can be proved that \( \lim_{a_s \to \infty} y = \frac{1}{2} L^2 \cdot (1 - \xi^2) \). In this case, Eq.4.40 gives \( s_f = \frac{1}{2} F \cdot a_s \cdot L^2 \cdot (1 - \xi^2) \). This is the case of zero interaction where no shear connection exists on the interface. In another extreme case of full interaction when \( a_s \) approaches infinity, \( y \) approaches zero giving \( s_f = 0 \);
- is directly proportional to the active slip strain coefficient \( a_s \);
- is directly proportional to the lateral force \( F \);
- increases monotonically with increase in \( L \). This relation is shown in Fig.30(b) in which \( \alpha = L \cdot \sqrt{a_s} \). When \( L \) increases, \( \alpha \) increases hence \( s_f \) \( (s_f = \frac{F \cdot a_s}{a_s} \cdot y) \) with \( y \) shown in Fig.30(b)) increases. However, it approaches an asymptote when \( L \) approaches infinity;
- is a maximum at the top of the column or \( \xi = 0 \), where the moment is zero. The maximum slip is given by
  \[
  s_{\text{max}} = F \cdot \frac{a_s}{a_s} \left[ \frac{1}{1 - \cosh(L \cdot \sqrt{a_s})} \right]
  \] (4.75)
and;
- decreases monotonically in a convex shape to zero at the support where the bending moment is a maximum.
(a). Function \[ y = \frac{1}{a_5} \left[ -\cosh(L \cdot \sqrt{a_3} \cdot \xi) / \cosh(L \cdot \sqrt{a_5}) \right] \]

(b). Function \[ y = 1 - \frac{\cosh(a \cdot \xi)}{\cosh(a)} \]

Fig. 30 Functions of Eq. 4.40

4.4.2 Slip Due To Axial Load

From Eq. 4.41 it can be seen that the slip caused by the axial load \( s_n \):

- decreases monotonically when the slip resistance ability coefficient \( a_5 \) increases, as shown in Fig. 31(a) by the function \[ y = \frac{1}{\sqrt{a_5}} \left[ e^{-L \cdot \sqrt{a_5}} - 2 \cosh(L \cdot \sqrt{a_5} \cdot \xi) / (e^{2L \cdot \sqrt{a_5}} + 1) \right] \] which is part of the slip term. Similar to \( s_f \), there are also two asymptotes for this function: \( \lim_{a_5 \to 0} y = L \cdot (1 - \xi) \) which gives \( s_n = N \cdot a_3 \cdot L \cdot (1 - \xi) \); and \( \lim_{a_5 \to \infty} y = 0 \) gives \( s_n = 0 \).
- is in direct proportion to the axial flexibility of element \( 1 \cdot a_j \);
- is in direct proportion to axial force \( N \);
- increases monotonically with increase in \( L \) at a point that keeps a fixed distance away from the top (\( L \) increases but \( x = \xi \cdot L = \text{constant} \)), as shown in Fig. 31(b), with an asymptote when \( L \) approaches infinity, i.e. \( \lim_{L \to \infty} s_n = \frac{N \cdot a_3}{\sqrt{a_5}} \cdot e^{-\sqrt{a_5}} \) where \( x \) is the distance from the top of the column to the point considered. However, at a point that has a relative position fixed (i.e. \( \xi = \text{constant} \) but \( \xi \neq 0 \), for example the point at the middle of the column where \( \xi = 0.5 \)), the slip initially increases with increase in \( L \), when \( L \) is less than a certain length \( L_c \). After this point \( L_c \), the slip
will reduce when the length of the member is further increased, as shown in Fig.31(c);
- is a maximum at the top. The maximum slip is given by
  \[ s_{\text{max}} = \frac{N \cdot a_5}{\sqrt{a_5}} \left[ 1 - \frac{2}{(e^{2(\xi L \sqrt{a_5})} - 1)} \right] \]  
  \[ (4.76) \]
  and;
- decreases monotonically to zero at the fixed end.

\[ (a) \quad y = \frac{1}{\sqrt{a_5}} \left[ e^{-(L \sqrt{a_5})} - 2 \cosh(L \sqrt{a_5}) \right] / (e^{2(L \sqrt{a_5})} + 1) \]

\[ (b) \quad y = e^{-L \sqrt{a_5}} - 2 \cosh(L \sqrt{a_5}) / (e^{2L \sqrt{a_5}} + 1) \]

\[ (c) \quad y = e^{-(L \sqrt{a_5})} - 2 \cosh(L \sqrt{a_5}) \sqrt{a_5} / (e^{2(L \sqrt{a_5})} + 1) \]

Fig.31 Functions of Eq.4.41

4.4.3 Slip Due To Boundary Slip

Equation 4.42 shows that the slip along the column induced by the boundary slip \( s_p \):
- decreases when the slip resistance ability coefficient \( a_5 \) increases. For the case of no interaction i.e. \( a_5 = 0 \), the slip is a constant value \( s_p \) along the length. For
case of approaching full interaction when \( a_s \) approaches infinity, the slip approaches zero except at the vicinity of the support where the slip is equal to \( s_p \) at the support;

- is in direct proportion to boundary slip \( s_p \); and

- reduces monotonically with the increase in length \( L \). Mathematically, it can be shown from Eq.4.42 that the slip at a point that has a certain distance away to the bottom of the column approaches an asymptote value when \( L \) approaches infinity; and

- monotonically increases from the top to the bottom of the column in a concave shape. The maximum slip value at the support when \( x=l \) is \( s_p \), and the minimum slip at the top is given by \( s_p = \frac{s_p}{\cosh(L \cdot \sqrt{a_s})} \).

5 DESIGN OF PLATING SYSTEM

The effectiveness of the new retrofitting system has been clearly shown in the previous sections. In this section, the problem of how to design the retrofitting system to satisfy a design requirement is studied. For a column, the ability to sustain certain level of lateral resistance after yielding until the required maximum lateral drift is important. Therefore, this concept is adopted as a guide to the retrofitting design in this work. The design procedure developed in this section is based on a specified target maximum lateral displacement, at which the retrofitted column is required to still work in its 'strength stiffened' region of a response curve as discussed in Section 3.2.2. In this way, the lateral resistance capacity and the integrity of the column can be assured at the target displacement. This design procedure is fundamentally the same to, and can be easily adapted for use with, the modern displacement based seismic design philosophy (Calvi and Kingsley, 1995; Moehle, 1996; Priestley, 1997 and 1998).

To develop the design procedure, several fundamental relations are needed which are firstly derived in Sections 5.1 and 5.2. The design procedures are then presented in Section 5.3.

5.1 GENERIC DEFORMATION - SLIP RELATION

To calculate the slip in the plastic hinge, a general relation between slip and deformation of the column is derived in this section.

The strains in the plate and the RC column at the interface can be expressed as

\[
\varepsilon_i = \kappa \cdot h_i + \varepsilon_{ic} \tag{4.77}
\]

\[
\varepsilon_3 = \varepsilon_{3c} - \kappa \cdot h_3 \tag{4.78}
\]

where \( \varepsilon_{ic} \) and \( \varepsilon_{3c} \) are the strains at the cross-sectional centroid of element 1 (RC column) and element 2 (plate), respectively, as shown in Fig.32.

Substituting Eqs.4.77 and 4.78 into Eq.4.11 and integrating with respect to \( x \) gives

\[
s = -(h_1 + h_3) \cdot \int \kappa \cdot dx + \int \varepsilon_{1c} \cdot dx - \int \varepsilon_{3c} \cdot dx + C \tag{4.79}
\]
\[ \therefore d\theta = -\kappa \cdot dx \]
\[ d\Delta_2 = -\varepsilon_{2c} \cdot dx \]
\[ d\Delta_1 = -\varepsilon_{1c} \cdot dx \]

where \( \theta \) is the rotation of the cross-section; \( \Delta_2 \) and \( \Delta_1 \) are the longitudinal displacements at the cross-sectional centroids of element 1 and 2, respectively.

Therefore, Eq. 4.79 is re-written as
\[ s(x) = (h_1 + h_2) \cdot \theta(x) + \Delta_1(x) - \Delta_2(x) + C \]
\[ \therefore |s|_{x=L} = 0; \quad \theta |_{x=L} = 0; \quad \Delta_2 |_{x=L} = 0; \quad \Delta_1 |_{x=L} = 0 \]
\[ \therefore C = 0 \]

which gives
\[ s = (h_1 + h_2) \cdot \theta + \Delta_1 - \Delta_2 \]
\[ (4.80) \]

The above relation shows that the slip at any cross-section is given by the rotation of that cross-section times the distance between centroids of element 1 and 2, plus the difference of axial shortening between element 1 and 2. This relation is general, because it is derived from the geometric relations with the only assumption being that plane sections remain plane for element 1 and 2 separately. Therefore, it is applicable for general non-linear analysis.

5.2 ULTIMATE PLASTIC HINGE ANALYSIS

The formulae related to the plastic hinge calculations that will be used in Section 5.3 are derived in this section. It must be noted that the derivations in this section are only applicable for monotonic loading because the stress of both concrete and steel cannot be determined by strain only under cyclic loading, as it is also dependent on the loading history. Therefore, all the analytical results that are related to stresses or forces are only applicable to monotonic loading conditions.

5.2.1 Slip in Plastic Hinge Region

The slip at the top of the plastic hinge is given by Eq.4.80 as
\[ s_p = (h_1 + h_2) \cdot \theta_p + \Delta_{tc} - \Delta_{2c} \]  \hspace{1cm} (4.81)

where \( \theta_p \) is the rotation of the cross-section on top of the plastic hinge, i.e. the total plastic hinge rotation; and \( \Delta_{tc} \) and \( \Delta_{2c} \) are the respective axial shortening of the plastic hinge at the centroids of element 1 and element 2. Each term of Eq.4.81 is discussed in the following paragraphs.

It is well documented in the literature that the lateral displacement of a member is mostly caused by the plastic hinge rotation at the ultimate load stage, as shown by the thin dotted line in Fig.24(a), i.e. \( \Delta \approx \Delta_p \). The displacement at the top of the column due to the plastic hinge is given by

\[ \Delta_p = \theta_p \cdot (L - 0.5 \cdot L_p) = \kappa_{cu} \cdot L_p \cdot (L - 0.5 \cdot L_p) \]  \hspace{1cm} (4.82)

where the ultimate curvature \( \kappa_{cu} \) is the curvature under ultimate displacement for cross-sections, which is assumed to be constant within the plastic hinge. The plastic hinge length of a cantilever column \( L_p \) can be estimated by the relation, obtained from extensive experimental studies by Priestley and Park (1987) and Paulay and Priestley (1992),

\[ L_p = 0.08L + 6d_a \]  \hspace{1cm} (4.83)

in which \( d_a \) is the diameter of the main reinforcement bar. Eq.4.82 was developed (Priestley and Park, 1987) by assuming that the plastic rotation \( \theta_p \) is concentrated at the center of the plastic hinge. In fact, this relation can be derived mathematically without this assumption by double integration of curvature along the length, or more easily, by the geometrical relation shown in Fig.33.

![Diagram](image)

From Eq.4.82, the ultimate curvature is given by

\[ \kappa_{cu} = \frac{\Delta_p}{(L - 0.5 \cdot L_p) \cdot L_p} \approx \frac{\Delta}{(L - 0.5 \cdot L_p) \cdot L_p} \]  \hspace{1cm} (4.84)

Therefore, \( \theta_p \) in the first term of Eq.4.81 can be calculated by

\[ \theta_p = \kappa_{cu} \cdot L_p \approx \frac{\Delta}{(L - 0.5 \cdot L_p)} \]  \hspace{1cm} (4.85)
The second and third terms in Eq.4.81 are given by
\[
\Delta_{lc} = \varepsilon_{lc} \cdot L_p
\]
\[
\Delta_{2c} = \varepsilon_{2c} \cdot L_p
\]
where \(\varepsilon_{lc}\) is given by
\[
\varepsilon_{lc} = \varepsilon_{cu} - \kappa_{cu} \cdot h_i
\]
in which \(\varepsilon_{cu}\) is the ultimate concrete strain on the compression face at which the RC member is considered failed. The strain of concrete when it is completely crushed (at zero strength) is considered as \(\varepsilon_{cu}\) in this work. This assumption sets a scene in which the ultimate limit state occurs: when the column achieves the required maximum drift, the concrete strain on the compression face reaches the ultimate strain \(\varepsilon_{cu}\).

There seems no evidence available in the literature to support the above assumption. However, from the mathematical and numerical studies (Wu, 2000), the point of complete concrete crush on the compression face, shown by '0' in Fig.33a, does provide a good indication to the end of the yield plateau for RC columns, after which rapid degradation occurs. The same trend can also been seen from Fig.11 for plated columns.

![Moment - Curvature Chart for N=1408KN (20%Nc)](image)

Fig.33a Moment-curvature relation of RC columns

To calculate \(\varepsilon_{2c}\), first the axial load \(N_2\) on the plate section is calculated, which is derived in detail in Section 5.2.2. With the axial force \(N_2\) and the curvature of the plate given by Eq.4.84, \(\varepsilon_{2c}\) can then be calculated with the formulae that are derived in Section 5.2.3.

### 5.2.2 Cross-Sectional Forces

When the concrete strain at the ultimate stage is \(\varepsilon_{cu}\) on the compression face or the stress is zero, the stress block is shown in Fig.34. If this stress profile is assumed to be triangular as shown in the shaded area, the axial force on the concrete is given by
\[
N_{conc} = \frac{1}{2} \cdot f_c \cdot \frac{\varepsilon_{cu}}{\kappa_{cu}} \cdot B
\]
where \(B\) is the width of the column and the remainder of the function of the right hand side \(f_{co}e_{cu}/2\kappa_{cu}\) is the area of the triangular stress block. When the stress block is not triangular, it can generally be written as

\[
N_{con} = \beta \cdot f_{co} \frac{e_{cu}}{\kappa_{cu}} \cdot B = \beta \cdot \frac{N_{e}}{\kappa_{cu}} 
\]

(4.90)

The above equation is similar to Eq.4.89 with the constant coefficient \(\frac{1}{2}\) replaced by a general coefficient \(\beta\). For a given stress-strain model, the shape of the stress block is defined from which \(\beta\) can be calculated by

\[
\beta = \frac{\int \sigma \cdot d\xi}{f_{co} \cdot x} 
\]

(4.90a)

where \(\int \sigma \cdot d\xi\) is the stress block area in which \(\xi\) is the distance of a point in the cross-section from the compression face, \(\sigma\) is the concrete stress at the corresponding point, and \(x\) is the depth of compression zone shown in Fig.34. If Mander’s stress-strain model of concrete is adopted (Mander et al. 1988a), the value \(\beta\) is found to be only a function of the concrete strength. This \(\beta\) value can then be calculated through numerical integration of the stress block. The results from numerical integration of Eq.(4.90a) are given in Fig.35.

![Fig.34. Concrete stress block](image)

![Fig.35 Area coefficient \(\beta\)](image)

The centroid of the stress block, which is needed to calculate the moment of the cross-section, is also a function of the concrete strength with Mander’s model. Its position, as shown in Fig.34, can be related to a parameter \(\delta\) by

\[
\delta = \frac{d}{x} 
\]

(4.91)
where \( d \) is the position of centroid from the compression face shown in Fig.34 that is given by
\[
d = \frac{\int \sigma \cdot \xi \cdot d\xi}{\int \sigma \cdot d\xi}
\]  
(4.91a)

The numerical integration of Eq.(4.91a) gives the value of \( \delta \) as shown in Fig.36.

![Graph showing the relationship between \( \delta \) and concrete strength](image)

**Fig.36 Centroid coefficient \( \delta \)**

For longitudinal reinforcing bars, the strains at the compression and tension sides are respectively
\[
\varepsilon_{sc} = \varepsilon_{cu} - \kappa_{cu} \cdot a
\]  
(4.92)
\[
\varepsilon_{su} = \varepsilon_{cu} - \kappa_{cu} \cdot (D - a)
\]  
(4.93)

where \( a \) is the concrete cover thickness from the face of the column to the center of reinforcing bar. Forces on the tension and compression reinforcement are respectively
\[
N_{su} = \begin{cases} 
\varepsilon_{su} \cdot E_s \cdot A_{su} & (\text{when } |\varepsilon_{su}| \leq \varepsilon_{sy}) \\
-f_{sy} \cdot A_{su} & (\text{when } \varepsilon_{su} \geq \varepsilon_{sy})
\end{cases}
\]  
(4.94)
\[
N_{sc} = \begin{cases} 
\varepsilon_{sc} \cdot E_s \cdot A_{sc} & (\text{when } |\varepsilon_{sc}| \leq \varepsilon_{sy}) \\
-f_{sy} \cdot A_{sc} & (\text{when } \varepsilon_{sc} \geq -\varepsilon_{sy})
\end{cases}
\]  
(4.95)

where \( \varepsilon_{sy} \) is the yield strain of reinforcing bar.

Therefore, the total force on the RC cross-section is
\[
N_1 = N_{conc} + N_{sc} + N_{su}
\]  
(4.96)

Hence the axial force on the plate can then be calculated from
\[
N_2 = N - N_1
\]  
(4.97)

### 5.2.3 Calculation Of The Plate Strain

With axial load on the steel plate (given by Eq.4.97) and the curvature of the plate at the column base (given by Eq.4.84) known, the strain at the cross-sectional centroid of the plate \( \varepsilon_{zc} \), which is required in Eq.4.87, can be easily calculated. The explicit form of \( \varepsilon_{zc} \) is derived in this section, assuming an elastic-plastic stress-strain model.
for the plate material. Based on different yielding conditions on the faces, it is discussed with the following 3 cases.

(1) Full elastic condition
When a compression axial load \( N_z \geq 0 \) is acting and when the compression face is linear elastic, the tension face must also be linear elastic. Therefore, the strain at the sectional centroid of the plate is simply given by

\[
\varepsilon_{2e} = \frac{N_z}{B \cdot t \cdot E_p}
\]  

(4.98)

where \( E_p \) is the elastic modulus of the plate.

(2) Compression face yielded and tension face elastic

\[
\begin{align*}
\varepsilon_{2e} &= \frac{N_z}{B \cdot t \cdot E_p} \\
\kappa &= \frac{E_p}{E_p - E_{py}}
\end{align*}
\]

\[
\varepsilon_{2e} = \kappa \cdot t / 2
\]

Fig. 37 Strain at centroid of plate

Figure 37 shows the stress and strain profiles of the plate in this case under the elastic-plastic stress-strain relation. The axial force in the plate can be calculated as

\[
N_z = B \left[ \left(\frac{t}{2} + \frac{\varepsilon_{2e} - \varepsilon_{py}}{\kappa}\right) \cdot f_{py} + \frac{1}{2} \cdot \frac{\varepsilon_{py}}{\kappa} \cdot f_{py} + \frac{1}{2} \cdot \left(\varepsilon_{2e} - \frac{t}{2} \cdot \kappa\right) \cdot E_p \cdot \left(\frac{t}{2} - \frac{\varepsilon_{2e}}{\kappa}\right) \right]
\]  

(4.99)

Substituting \( f_{py} = \varepsilon_{py} \cdot E_p \) into Eq.4.99 and re-arranging gives

\[
\varepsilon_{2e}^2 - (2\varepsilon_{py} + \kappa \cdot t) \cdot \varepsilon_{2e} + \frac{\kappa^2 \cdot t^2}{4} - (\kappa \cdot t - \varepsilon_{py}) \cdot \varepsilon_{py} + \frac{2\kappa \cdot N_z}{B \cdot E_p} = 0
\]  

(4.100)

Equation 4.100 has two solutions

\[
\varepsilon_{2e} = \frac{2\varepsilon_{py} + \kappa \cdot t - \sqrt{(2\varepsilon_{py} + \kappa \cdot t)^2 - 4 \cdot \left(\frac{\kappa^2 \cdot t^2}{4} - (\kappa \cdot t - \varepsilon_{py}) \cdot \varepsilon_{py} + \frac{2\kappa \cdot N_z}{B \cdot E_p}\right)}}{2}
\]

\[
= \varepsilon_{py} + \frac{1}{2} \cdot \kappa \cdot t - \Delta \varepsilon
\]  

(4.101)

and
\[ \varepsilon_{2c} = \frac{2 \varepsilon_{py} + \kappa \cdot t + \sqrt{(2 \varepsilon_{py} + \kappa \cdot t)^2 - 4 \left[ \frac{\kappa^2 \cdot t^2}{4} - (\kappa \cdot t - \varepsilon_{py}) \cdot \varepsilon_{py} + \frac{2 \kappa \cdot N_2}{B \cdot E_p} \right]}}{2} \]

\[ = \varepsilon_{py} + \frac{1}{2} \cdot \kappa \cdot t + \Delta \varepsilon \]  \hspace{1cm} (4.102)

where \( \Delta \varepsilon \) is a positive term given by

\[ \Delta \varepsilon = \frac{1}{2} \cdot \kappa \cdot t + \sqrt{(2 \varepsilon_{py} + \kappa \cdot t)^2 - 4 \left[ \frac{\kappa^2 \cdot t^2}{4} - (\kappa \cdot t - \varepsilon_{py}) \cdot \varepsilon_{py} + \frac{2 \kappa \cdot N_2}{B \cdot E_p} \right]} \]  \hspace{1cm} (4.103)

One of the two solutions above is false and can be found out through the following calculation. The strain at the tensile face is given by

\[ \varepsilon_i = \varepsilon_{2c} - \frac{1}{2} \cdot \kappa \cdot t \]  \hspace{1cm} (4.104)

Substituting Eq.4.102 into Eq.4.104 gives

\[ \varepsilon_i = \varepsilon_{py} + \Delta \varepsilon \geq \varepsilon_{py} \]  \hspace{1cm} (4.105)

Equation 4.105 means that with the solution of Eq.4.102 the whole cross-section yields for any given axial load and curvature. Obviously it is not the correct solution and should be discarded. Therefore, only Eq.4.101 is the correct solution.

The solution given by Eq.4.101 includes two special cases.

(a) The first one is when the tension face just yields in compression or the whole cross-section just yields in compression. The conditions for this to occur can be derived as follows.

When \( \Delta \varepsilon = 0 \), from Eq.4.105, the strain at the tensile face is \( \varepsilon_i = \varepsilon_{py} \) or just yields. Based on Eq.4.103, \( \Delta \varepsilon = 0 \) necessitates

\[ (2 \varepsilon_{py} + \kappa \cdot t)^2 - 4 \left[ \frac{\kappa^2 \cdot t^2}{4} - (\kappa \cdot t - \varepsilon_{py}) \cdot \varepsilon_{py} + \frac{2 \kappa \cdot N_2}{B \cdot E_p} \right] = 0 \]  \hspace{1cm} (4.106)

Therefore, Eq.4.106 gives the condition (relation between \( \kappa \) and \( N_2 \)) at which the whole cross-section just yields in compression.

When the left-hand side of Eq.4.106 gives a negative solution, or when \( \Delta \varepsilon < 0 \), i.e.

\[ (2 \varepsilon_{py} + \kappa \cdot t)^2 - 4 \left[ \frac{\kappa^2 \cdot t^2}{4} - (\kappa \cdot t - \varepsilon_{py}) \cdot \varepsilon_{py} + \frac{2 \kappa \cdot N_2}{B \cdot E_p} \right] < 0 \]  \hspace{1cm} (4.107)

Equation 4.100 has no solution. Physically, this case occurs when the balance of axial forces in the cross-section cannot be achieved even after the whole cross-section yields. In this case, the plate thickness is too small to sustain the axial force and must be increased.

(b) The second special case occurs when the tension face just yields in tension. The condition for this to occur is

\[ \varepsilon_{2c} = \frac{1}{2} \cdot \kappa \cdot t - \varepsilon_{py} \]  \hspace{1cm} (4.108)

because substituting Eq.4.108 into Eq.4.104 gives \( \varepsilon_i = -\varepsilon_{py} \). This means the tension face just yields in tension.
If Eq. 4.108 becomes \( \varepsilon_{2c} < \frac{1}{2} \cdot \kappa \cdot t - \varepsilon_{py} \), Eq. 4.104 gives \( \varepsilon_t < -\varepsilon_{py} \). In other word, the tension face has already yielded in tension. In this case, the stress distribution shown by Fig. 37, which is used to drive Eq. 4.101, is incorrect and must be considered as the third case discussed below.

(3) Compression face yielded and tension face yielded in tension

In this case, the stress distribution is changed to that shown in Fig. 38.

Fig. 38 Profiles when tension side yielded in tension

The axial force on the cross-section is given by

\[
N_2 = B \left[ \left( \frac{t}{2} + \frac{\varepsilon_{2c} - \varepsilon_{py}}{\kappa} \right) \cdot f_{py} - \left( \frac{t}{2} - \frac{\varepsilon_{2c} + \varepsilon_{py}}{\kappa} \right) \cdot f_{py} \right]
\]  

(4.109)

which gives the solution

\[
\varepsilon_{2c} = \frac{N_2 \cdot \kappa}{2B \cdot f_{py}}
\]  

(4.110)

5.3 DISPLACEMENT BASED PLATING DESIGN PROCEDURE

With all the necessary formulae derived in Sections 5.1 and 5.2, the design of the plating system is straightforward. It is outlined in the following 7 steps.

1. Calculate the ultimate curvature at the plastic hinge based on the target maximum displacement, or drift ratio.

   When the design displacement or inter-storey drift ratio \( \theta (\theta = \Delta / L) \) is given, \( \Delta = \theta \cdot L \). Assuming that the plastic deformation is concentrated in the plastic hinge and the elastic deformation is relatively small compared to the plastic deformation, Eqs. 4.83 and 4.84 are used to calculate the ultimate curvature \( \kappa_{cu} \).

2. For the cross-section at the plastic hinge location, calculate the following when the ultimate curvature is reached:
   i) The axial force on concrete using Eq. 4.90 and Fig. 35;
ii) The axial forces in the reinforcement using Eqs.4.92-4.95; and
iii) The axial force in the plate $N_2$ by Eqs.4.96 and 4.97

3. Estimate steel plate thickness $t$.
   The estimation can be based on the full yield thickness $t_{\text{min}} = N_2 / (f_{py} \cdot B)$ taking into account an allowance (e.g. a factor of 1.2) for non-uniform stress distribution in the cross-section of the plate; e.g.
   \[
   t = 1.2 \cdot \frac{N_2}{B \cdot f_{py}}
   \]  
   (4.111)
   This is only a first approximation and could be adjusted from calculations in step 4.

4. Assuming the first bolt is immediately above the plastic hinge, calculate the slip $s_p$ of this bolt by Eq.4.81. The three terms in Eq.4.81 are calculated using Eq.4.85, Eqs.4.86 and 4.88, and Eq.4.87, respectively. Formulae derived in Section 5.2.3 are needed to calculate the strain term $\varepsilon_{2e}$ in Eq.4.87. The calculation of $\varepsilon_{2e}$ will indicate whether the plate thickness assumed in step 3 is adequate or not. Adjust $t$ if it is not adequate.

5. Calculate the force on the first bolt.
   Based on the slip $s_p$ calculated in step 4 and the slip-shear load relation as shown by the typical curve of Fig.39, the force at the first bolt can be calculated.

(b) Experimental load-slip relation (Teh et al. 1999)
Fig.39 Load-slip relation of an anchor bolt
The slip-shear load relation is affected not only by the properties of bolt but also by the properties of the concrete and the plate. It is currently obtained from bolt shear test. That is to say a bolts shear test must be conducted to get this relation for a specific bolt and concrete combination. Extensive experimental tests can provide general guides on the relations.

When \( s_p \) is less than the elastic limit (from point A to B in Fig.39(b)), a simple linear relation between slip and shear load on the bolt can be assumed, i.e. \( F_s = s_p \cdot K_n \). (The bolt is usually working in the elastic range, because the limit in the ultimate compressive concrete strain \( \varepsilon_{cu} \) does not allow large slips to occur.)

6. Estimate the required number of bolts.
As a first approximation, assume the slip is uniformly distributed along its length. This assumption is usually a good approximation at the ultimate limit stage, as can be seen from Figs.22, 23 and 29. Therefore, the shear forces is the same in all the bolts. Based on Eq.4.1, the number of bolts required is then given by

\[
n = \frac{N_2}{F_b} \tag{4.112}
\]

This estimation is usually adequate for design purpose.

7. Adjust bolts number.
If higher precision is required or the distribution of slip cannot be reasonably considered uniform at the ultimate limit stage, the result given in step 6 can be modified. First, calculate the bolt spacing based on the previously estimated number of bolts. Second, calculate the moment of the bottom cross-section based on the forces calculated in step 2, which determines the lateral force applied on top of the column. Eq.4.91 and Fig.36 are needed to calculate the moment due to the concrete. The slip distribution is then calculated from linear elastic theory using Eqs.4.39-4.42 based on the lateral force, axial load, bolt spacing and slip \( s_p \) calculated previously. The slip at each bolt position can then be calculated to get the shear force on each bolt. The summation of all bolt forces gives the axial force in the steel plate at the plastic hinge. If this plate force is close enough to the value of \( N_2 \) that was calculated in step 2, the bolt design is adequate. Otherwise adjust bolts number and spacing and repeat step 7 until the required precision is achieved.

5.4 EXAMPLE
The column in Fig.6 is to be designed for an axial load of 360 kN and a maximum inter-story drift ratio of \( \theta = \Delta / L = 2.5\% \) or \( \Delta = 30\text{mm} \). The geometric and material properties are the same as those given in Section 3.1.

From Eq.4.83, \( L_p = 0.08L + 6d_t = 192 \text{ (mm)}, \) say 200mm.

Eq.4.84 gives \( \kappa_{cu} = \frac{\Delta}{(L - 0.5 \cdot L_p) \cdot L_p} = 1.36 \times 10^{-4} \text{ (mm)}^{-1} \).

From Fig.35, \( \beta = 0.61, \) and by Eq.4.90

\[
N_{conc} = \beta \cdot f_{ce} \cdot \frac{\varepsilon_{cu}}{\kappa_{cu}} ; B = 215 \text{ (kN)}.
\]

By Eqs.4.92 and 4.93, \( \varepsilon_{uc} = \varepsilon_{cu} - \kappa_{cu} \cdot a = 0.00115 < \varepsilon_{fy}, \)

\( \varepsilon_{u} = \varepsilon_{cu} - \kappa_{cu} \cdot (D - a) = -0.016 < -\varepsilon_{fy}, \) yielded in tension,
\[ \therefore N_{sc} = A_{sc} \cdot E_s \cdot \varepsilon_{sc} = 92.5 \text{ (kN)}, \ N_{st} = A_{st} \cdot f_{sp} = -219.9 \text{ (kN)}. \]

From Eqs. 4.96 and 4.97, \[ N_2 = N - N_{conc} - N_{sc} - N_{st} = 272 \text{ (kN)}. \]

The minimum steel plate thickness is \[ N_2 / (B \cdot f_{py}) = 5.4 \text{ (mm)}, \] so a plate thickness of 6 mm is chosen.

The first bolt is placed 200 mm \((L_p = 200 \text{ mm})\) above the bottom of the column. To calculate the slip of the first bolt, the following are calculated:

By Eq. 4.85, \[ \theta_p = \kappa_{cu} \cdot L_p = 0.0272; \]

From Eqs. 4.86 and 4.88, \[ \Delta_{ic} = (\varepsilon_{cu} - \kappa_{cu} \cdot h_1) \cdot L_p = -1.52 \text{ (mm)}; \]

Using Eq. 4.101 to calculate \( \varepsilon_{2c} \),

\[
\varepsilon_{2c} = \frac{2\varepsilon_{py} + \kappa_{cu} \cdot \tau - \sqrt{(2\varepsilon_{py} + \kappa_{cu} \cdot \tau)^2 - 4 \cdot \left[ \frac{\kappa_{cu}^2 \cdot \tau^2}{4} - (\kappa_{cu} \cdot \tau - \varepsilon_{py}) \cdot \varepsilon_{py} + \frac{2\kappa_{cu} \cdot N_2}{B \cdot E_p} \right]}}{2}
\]

= 0.00124, checking with Eq. 4.106 and Eq. 4.108 indicates that the tension side of the plate is neither yielded in tension or yielded in compression. This confirms that the conditions to use Eq. 4.101 is satisfied. It also suggests that the selected plate thickness is adequate;

By Eq. 4.87, \[ \Delta_{2c} = \varepsilon_{2c} \cdot L_p = 0.25 \text{ (mm)}; \]

From Eq. 4.81, \[ s_p = (h_1 + h_2) \cdot \theta_p + \Delta_{ic} - \Delta_{2c} = 1.03 \text{ (mm)}. \]

Assuming the same slip in the remaining bolts, the force at each bolt is \[ F_b = s_p \cdot K_b = 23.7 \text{ (kN)}, \] and therefore the number of bolts required from Eq. 4.112 is

\[ n = \frac{N_2}{F_b} = 12, \text{ or 6 rows of two}. \]

Comparing the above design results to the column in Fig. 6 analysed by the non-linear numerical computer program. The numerical results in Fig. 7 gives the lateral displacement \( \Delta = 33.9 \text{ mm} \) when the strain at the extreme compressive fibre of the concrete equals to 0.006. This displacement 33.9 mm includes the elastic deformation above the plastic hinge which is calculated to be 4.4 mm. Therefore the lateral displacement due to the plastic hinge rotation only is 29.5 mm which is very close to the specified design displacement of 30 mm. The slip at the first bolt is calculated to be 1.034 mm in the above design that is also very close to the result of 1.042 given by the non-linear computer program, which is shown in Fig. 22.
6 SUMMARY

Composite plating rectangular RC columns by bolting steel or FRP plates to their compression and tension surfaces improves column ductility through partial-interaction between the plate and the column. The essence of this technique of composite partial interaction plating columns is that it increases the capacity of the compression face without significantly increasing the capacity of the tension face. This increases the compressive resistance of the RC column and delays concrete crushing which in turn improves the section ductility. Non-linear partial-interaction computer simulations have been used to illustrate the beneficial effects of this new retrofitting technique. Mathematical analyses have also been carried out to better understand the fundamentals of the plated columns as well as to derive design formulae. A displacement based design procedure that is developed to design the details of the plating system has also been outlined in this report.
7 REFERENCES


8 NOTATION

\( A_g \) = gross cross-sectional area of column

\( A_i \) = cross-sectional area of longitudinal reinforcement

\( A_r \) = cross-sectional area of longitudinal reinforcement at compressive side

\( A_u \) = cross-sectional area of longitudinal reinforcement at tensile side

\( a \) = thickness of concrete cover to center of longitudinal reinforcement bar

\( a_0 \) = \( \frac{L}{K_s} \), unit longitudinal flexibility of shear connector/bolt

\( a_1 \) = \( \frac{Ei}{EA \cdot \sum EI} \), passive slip strain coefficient

\( a_2 \) = \( \frac{(h_1 + h_2)}{\sum EI} \), active slip strain coefficient

\( a_3 \) = \( \frac{1}{(EA)_1} \), axial flexibility of element 1

\( a_4 \) = \( \frac{(h_1 + h_2) \cdot EA}{EI} \), unit slip curvature

\( a_5 \) = \( \frac{a_1}{a_0} \), slip resistance ability coefficient

B = breadth of column section

C = constant of integration

D = depth or diameter of cross-section

d = distance from centroid of stress block to compression face

\( d_s \) = diameter of longitudinal reinforcement

E = modulus of elasticity

\( E_c \) = modulus of elasticity of concrete

\( E_h \) = strain hardening stiffness of steel bar

\( E_p \) = modulus of elasticity of plate

\( E_{ph} \) = strain hardening stiffness of plate

\( E_s \) = modulus of elasticity of steel bar

\( \overline{EA} \) = defined as \( \frac{1}{EA} = \frac{1}{(E \cdot A)_1} + \frac{1}{(E \cdot A)_2} \)

\((EA)_1\) = axial rigidity of element 1

\((EA)_2\) = axial rigidity of element 2

\( \overline{EI} \) = \( \sum EI + EA \cdot (h_1 + h_2)^2 \)

\((EI)_1\) = flexural rigidity of element 1

\((EI)_2\) = flexural rigidity of element 2

\( \sum EI \) = \((E \cdot I)_1 + (E \cdot I)_2 \)

\( e_c \) = eccentricity of compressive resultant

\( e_t \) = eccentricity of tensile resultant
\( F \) = lateral force applied at top of cantilever column
\( F_b \) = bolt shear force
\( F_{by} \) = yield strength of bolt in shear
\( F_f \) = longitudinal shear force on shear connectors in shear span of length \( x \) from top of column due to lateral force \( F \) only
\( F_n \) = longitudinal shear force on shear connectors in shear span of length \( x \) from top of column due to axial force \( N \) only
\( F_{sh} \) = \( F_f + F_n + F_{sy} \), total longitudinal shear force on shear connectors in shear span of length \( x \) from top of column
\( F_{sy} \) = longitudinal shear force on shear connectors in shear span of length \( x \) from top of column due to non-zero boundary slip \( s_p \) only
\( f_{co} \) = compressive strength (peak stress) of unconfined confined concrete
\( f_{cc} \) = compressive strength of confined concrete
\( f_{ct} \) = tensile strength of concrete
\( f_{sy} \) = yield strength of stirrups
\( f_{py} \) = yield strength of plate
\( f_{oy} \) = yield strength of steel bar
\( h_1 \) = distance from centroid of element 1 to interface
\( h_2 \) = distance from centroid of element 2 to interface
\( I \) = second moment of area
\( K_s \) = elastic shear stiffness of bolt
\( K_{sh} \) = shear strain hardening stiffness of bolt
\( L \) = length of cantilever column
\( L_e \) = length of elastic part of member, \( L_e = L - L_p \)
\( L_i \) = length from top of column to section \( i \) (\( i = 0-\infty \))
\( L_p \) = plastic hinge length of column
\( L_s \) = longitudinal spacing of bolts
\( M \) = moment of section
\( M_0 \) = moment at section where lateral load is applied
\( M_1 \) = moment due to element 1
\( M_2 \) = moment due to element 2
\( M_i \) = moment at section \( i \) (\( i = 1-\infty \))
\( N \) = axial load of column or total axial force in a cross-section
\( N_{i1} \) = axial force on element 1
\( N_{i2} \) = axial force on element 2
\( N_c = f_{co} \cdot B \cdot D \), axial crush load of concrete section
\( N_{concr} \) = axial force on concrete
\( N_cr \) = compressive resultant in cross-section
\( N_{pl} \) = axial force on plate
\( N_{ax} \) = axial force in compressive reinforcement bar
\( N_{st} \) = axial force in tensile reinforcement bar
\( n \) = number of bolts
\( p \) = intensity of distributed load
\( Q \) = shear force in cross-section
\( Q_1 \) = shear force on element 1
\( Q_2 \) = shear force on element 2
\( q \) = longitudinal shear force per unit length or shear flow
\( R_c \) = experimentally determined material constant for Menegotto-Pinto model
\( R_s \) = slip radius
\( r \) = radius of curvature; or \( r = \frac{E_c}{E_c - E_{sec}} \)
\( s \) = longitudinal slip of bolt; or coordinate along length; or longitudinal center to center spacing of stirrups
\( s_f \) = slip term due to lateral force F
\( s_a \) = slip term due to axial force N
\( s_p \) = slip at support of elastic member or top of plastic hinge
\( s_{sp} \) = slip term due to non-zero boundary slip \( s_p \)
\( t \) = depth of element 1 or thickness of plate
\( v \) = lateral deflection of column
\( v_{f_{full}} \) = lateral deflection induced by lateral force F with full interaction
\( v_{a_{full}} \) = lateral deflection induced by axial load N with full interaction
\( x \) = depth of neutral axis in a cross-section; or distance along beam; or \( x = \frac{E_c}{E_{cc}} \); or some intermediate variable
\( y \) = y coordinate; functions
\( y_1 \) = y coordinate of focal point 1
\( y_2 \) = y coordinate of focal point 2
\( a \) = \( L \cdot \sqrt{a_s} \)
\( a_1 \) = experimentally determined material constant for Menegotto-Pinto model
\( a_2 \) = experimentally determined material constant for Menegotto-Pinto model
\( \beta \) = area coefficient of stress block
\( \Delta \) = lateral displacement of cantilever column
\( \Delta_1 \) = longitudinal displacement of element 1 or RC column at centroid of cross-section
\( \Delta_{tc} \) = axial shortening of element 1 at centroid
\( \Delta_2 \) = longitudinal displacement of element 2 or plate at centroid of cross-section
\( \Delta_{2\sigma} \) = axial shortening of element 2 at centroid
\( \Delta_e \) = lateral displacement at top of column due to elastic deformation above plastic hinge
\( \Delta_{t_1} \) = lateral displacement of column at section i (i=0-n)
\( \Delta_p \) = lateral displacement at top of column due to plastic hinge rotation only
\( \Delta_y \) = lateral displacement at first yield of tensile reinforcement
\( \Delta_u \) = ultimate lateral displacement when lateral force equals to 80% of lateral force at yield point
\( \delta \) = \( d/x \), centroidal relative height
\( \varepsilon \) = strain
\( \varepsilon_1 \) = strain of element 1 at interface
\( \varepsilon_{1c} \) = strain at centroid of cross-section of element 1 or RC column
\( \varepsilon_2 \) = strain of element 2 at interface
\( \varepsilon_{2c} \) = strain at centroid of cross-section of element 2 or plate
\( \varepsilon_{co} \) = strain at maximum stress \( f_{co} \) of unconfined concrete
\( \varepsilon_{cu} \) = ultimate compressive strain of concrete
\( \varepsilon_s \) = steel strain
\( \varepsilon_{sc} \) = strain of reinforcement bar at compressive side
\( \varepsilon_{slp} \) = slip strain
\( \varepsilon_u \) = strain of reinforcement bar at tensile side
\( \varepsilon_r \) = tensile rupture strain of concrete = \( f_t/E_c \); strain at tension face
\( \varepsilon_y \) = yield strain of steel
\( \kappa \) = curvature of section
\( \kappa_{cu} \) = ultimate curvature or curvature of section when extreme compressive fiber of concrete reaches ultimate strain \( \varepsilon_{cu} \)
\( \kappa_i \) = curvature of section \( i \) (\( i=1\sim n \))
\( \mu \) = Poisson’s ratio; ductility factor
\( \theta \) = rotation of a cross-section; inter-storey drift ratio
\( \theta_p \) = rotation of plastic hinge
\( \zeta \) = \( x/L \), normalized coordinate \( x \); variable of integration; plastic strain of last excursion when used in Menegotto-Pinto model

**Symbols:**
\( \Delta \) = point when concrete on the tension face first cracks
\( \times \) = where \( f_{co} \) at \( \varepsilon_{co} \) attained at compression face of concrete
\( + \) = onset of yielding of tension reinforcing bars
\( \star \) = onset of yielding of compression reinforcing bars
\( \diamond \) = crushing of concrete at \( \varepsilon_{cu} \) at compression face
\( - \) = crushing of concrete at \( \varepsilon_{cu} \) adjacent to compression reinforcing bars
\( \bullet \) = onset of yielding of one bolt
\( \bullet \) = yielding of all bolts
\( \triangle \) = yielding of plate in the whole bottom cross-section
\( \blacklozenge \) = tensile yielding of compression reinforcing bars
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