Step-by-step analysis of creep and shrinkage effects in prestressed concrete

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ABSTRACT: Creep and shrinkage play a very significant role in the long-term behaviour of prestressed concrete members, particularly with regard to loss of prestress and long-term deflections, and their effects must be allowed for in design. Although simple approximations can sometimes be used in routine design calculations, a more basic approach is often needed. This report describes a step-by-step method of analysis which provides a reliable and reasonably accurate basis for investigating creep and shrinkage effects in prestressed flexural members.

Key Words: prestressed concrete, design, analysis, beams, creep, shrinkage warping,
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1. Introduction

Various methods are available to designers to evaluate time effects in prestressed concrete members. These range from simple empirical approximations through to sophisticated and complex viscoelastic analyses. Basic concepts of creep and shrinkage in concrete and methods of analysis have been described by Warner et al. (1988), and by Gilbert et al. (2011). A thorough review of the alternative approaches to creep and shrinkage analysis has been provided by Bazant (1982, 1988).

The age-adjusted effective modulus is probably the simplest method for estimating creep effects in prestressed concrete members. However, it gives only very approximate results and can be unreliable when applied to complex situations. The step-by-step method of analysis explained in this report is a more general and more reliable approach for investigating time-varying behaviour, and can take account of shrinkage and temperature effects, as well as creep.

In comparison with the more complex viscoelastic methods of analysis, the step-by-step method has an advantage in that it uses only the simple but fundamental structural concepts of equilibrium, compatibility and short-term elastic response. It is therefore relatively easy to understand and to apply to specific problems, while giving reasonably accurate results.

In this report the step-by-step method is used to analyse the effects of creep and shrinkage in uncracked prestressed concrete sections. To introduce the method, it is first used to evaluate creep effects in a simple concrete cross-section with axial prestress. It is then applied to progressively more complex cases. The different cross-sections considered in the report are shown in Figure 1. Various numerical examples are included which show how, and to what extent, creep and shrinkage affect the long-term behaviour of prestressed concrete members.

Although the step-by-step method is only used here to analyse creep and shrinkage effects in uncracked prestressed concrete sections, it can also be applied to cracked cross-sections and indeed to entire members and structural systems. In the analysis of a cracked cross-section, the neutral axis will usually change over time, and stresses and strains have to be evaluated in both the compressive concrete above the neutral axis and in the tensile cracked regions in each time step. This can be accomplished by dividing the cross-section into many thin horizontal layers of concrete, with additional layers of steel and tendon as appropriate to represent the tendon and reinforcement. To treat biaxial bending the section can be divided into small concrete fibres, plus fibres of steel and tendon (Warner, 1969).

The time-varying behaviour of an entire prestressed member can be analysed by treating it as a large number of small inter-connected segments, and by undertaking a step-by-step analysis for the cross-section of each segment. Average curvatures in the segments are integrated to calculate flexural deflections. Allowance can also be made for progressive bond breakdown in local regions around the flexural cracks (Rebentrost, et al, 2002).

An indeterminate structural system can also be analysed. However, if non-linear effects are allowed for, the analysis becomes, in effect, a step-by-step computer simulation of time-varying structural behaviour (Kawano et al, 1996). Software packages are gradually becoming available to allow this type of computer analysis to be used in the design office to predict the time-varying behaviour of complex members and structures. Some packages can also take account of progressive changes in the applied loads and in the structural system itself over time. Changing loads and a changing structural system have to be allowed for in design when, for example, a prestressed concrete bridge is to be constructed using the launching method or the balanced cantilever method.

It needs to be emphasised here that even the most sophisticated methods of analysis cannot be relied on to give accurate predictions of structural behaviour. Any theoretical analysis needs to be
used with a proper understanding of its limitations, and also with the best relevant available material test data. This is especially the case when creep and shrinkage become important issues in pre-stressed concrete design.

2. Step-by-step analysis

We are restricting our attention to the analysis of time effects in uncracked prestressed concrete cross-sections. Irrespective of the type of cross-section, the analysis follows a simple sequence of steps that begins with an elastic analysis at time \( t_0 \) to determine the conditions at the start of the process when stress is first applied to the concrete. The progressive changes that subsequently occur are evaluated in discrete time steps.

During each time step, the concrete is first allowed to shrink and to creep freely under the conditions existing when the step begins. These initial conditions will have been determined from the analysis of the previous time step, or, for the first time step, from the initial elastic analysis. To allow free creep and shrinkage to occur, the concrete is assumed not to be bonded to any adjacent reinforcing steel and tendon, so that strain compatibility between the concrete and the steel and tendon is progressively lost. At the end of the time step, compatibility of strains is restored by applying an equilibrating force system to the materials. An elastic analysis is used to evaluate this force system, which typically consists of a tensile force decrement \( \Delta X_c \) applied to the concrete and an equal but opposite compressive force decrement applied jointly to the tendon and any reinforcing steel present. The elastic analysis determines the new stresses and strains in the concrete, tendon and steel at the end of the time step, and hence the initial conditions for the next time step.

The time steps are \( (t_0 \text{ to } t_1), (t_1 \text{ to } t_2), \text{ etc up to } (t_{n-1} \text{ to } t_n) \). These are best chosen so that about the same amount of free inelastic strain occurs in each step, which means that the steps must vary in size. Usually the last time point, \( t_n \), will be at time infinity, \( t^* \), so that the final step, \( (t_{n-1} \text{ to } t_n) \), is infinitely large. The choice of time steps must also allow for any changes in sustained load over time. The number of time steps, \( n \), will usually be quite small. Indeed, a one-step analysis will be sufficiently accurate for many practical calculations.

Except for some very simple cases, such as for the sections in Figures 1(a) and 1(b), stress and strain gradients exist across the section, and inelastic and elastic increments in the strain gradient have to be determined for each time step.

In this report we will consider the effects of creep and shrinkage separately, because this conforms with the approach to deflection calculations stipulated by the Australian Concrete Structures Standard, AS 3600-2009. It is of course possible, and may often be advantageous, to analyse a section for the simultaneous effects of creep and shrinkage.

3. Creep in an unreinforced, axially prestressed member

We first consider the simple case of creep in the axially prestressed cross-section shown in Figure 1(a). The analysis is relatively simple because there is no strain gradient over the cross-section, and only a single typical concrete fibre therefore has to be considered. This means that only one equilibrium equation is used in the elastic analysis at the end of each time step, namely that the
decrement in tensile force in the tendon is equal to the decrement in compressive force in the concrete: \( \Delta X_p = \Delta X_c \).

An initial elastic analysis gives the stress in the concrete at the start of the process, \( \sigma_{\text{co}} \), and the stress in the tendon, \( \sigma_{\text{po}} \). In the first time step, from \( t_0 \) to \( t_1 \), the concrete creeps freely under the assumed constant stress \( \sigma_{\text{co}} \), while the stress and strain in the tendon remain constant. The creep strain increment is \( \Delta \varepsilon_{\text{cc}1} \). At the end of the time step, \( t_1 \), compatibility is restored by applying a tensile force \( \Delta X_{c1} \) to the concrete and an equal compressive force \( \Delta X_{p1} \) to the tendon. An elastic analysis is used to evaluate these force increments and hence the elastic decrement in the tendon strain, \( \Delta \varepsilon_{\text{p1}} \), the decrement in the concrete compressive stress \( \Delta \sigma_{\text{c1}} = \Delta X_c / A_c \), and the corresponding elastic strain decrement \( \Delta \varepsilon_{\text{ce}1} \).

Free creep occurs in the second time step under both the original compressive stress \( \sigma_{\text{co}} \) and the small tensile stress \( \sigma_{\text{c1}} \). At the end of the second step, compatibility is restored by introducing a compressive force increment \( \Delta X_{p2} \) to the tendon and an equal tensile force \( \Delta X_{c2} \) to the concrete, which induces tensile stress and elastic strain increments in the concrete, \( \Delta \sigma_{\text{c2}} \) and \( \Delta \varepsilon_{\text{ce}2} \), with a corresponding stress change \( \Delta \sigma_{\text{p2}} \) in the tendon. A two-part analysis is thus carried out for each time step.

At the end of the last time step (usually at \( t^n \)) the final concrete stress is the initial compressive stress minus the tensile stress increments applied at the end of each time step:

\[
\sigma_{\text{c}}^* = \sigma_{\text{co}} - \Delta \sigma_{\text{c1}} - \Delta \sigma_{\text{c2}} - \ldots - \Delta \sigma_{\text{en}}
\]  
(EQ 1)

Similarly, the final elastic strain is:

\[
\varepsilon_{\text{ce}}^* = \varepsilon_{\text{ceo}} - \Delta \varepsilon_{\text{ce}1} - \Delta \varepsilon_{\text{ce}2} - \ldots - \Delta \varepsilon_{\text{cen}}
\]  
(EQ 2)

The total creep strain at the end of the process is the sum of the increments in each time step:

\[
\varepsilon_{\text{ce}}^* = \Delta \varepsilon_{\text{ce}1} + \Delta \varepsilon_{\text{ce}2} + \ldots + \Delta \varepsilon_{\text{cen}}
\]  
(EQ 3)

The final total strain in the concrete is \( \varepsilon_{\text{c}}^* = \varepsilon_{\text{ce}}^* + \varepsilon_{\text{cc}}^* \).

In the numerical examples that follow, a one-step analysis (Example 1) and a three-step analysis (Example 2) are used, and the results are compared. It will be seen that the change from one step to three steps produces only a marginal change in the numerical results, but requires a very substantial increase in computational effort.

**EXAMPLE 1: ONE-STEP ANALYSIS, SYMMETRIC SECTION, NO REINFORCEMENT**

The one-step analysis is applied to a typical cross-section in an axially prestressed member of original length \( L \), as shown in Figure 2. The left end is assumed to be fixed; the right end, which is free to move, is originally located at AA'. Due to prestressing at time \( t_0 \) the member shortens eas-
tically by $\varepsilon_{\text{cee}} L$ and the right end moves to BB'. At time $t^*$, after free creep strain of $\varepsilon_{\text{cc}}^*$ has occurred, the concrete has shortened by a further $\varepsilon_{\text{ccc}}^*$ and the right end has moved to CC'. During free creep, the right end of the tendon remains at BB'. For compatibility to be restored at time $t^*$, the right end of the concrete has to move back from CC' by the amount $\Delta e_{\text{cc}}^*$ to DD', while the tendon has to shorten by $\Delta e_{\text{p}}$ so that its end moves from BB' to DD'. As we are using only a single time step, the subscript "1" can be omitted from the various increments in force, strain and stress that occur during the time step.

In this numerical example, the concrete section of Figure 1(a) is taken to be 400 mm square, with the initial prestressing force $P_o = 1600$ kN, applied at 14 days. The following data apply:

\[ \varphi(t^*, 14) = 3.0; \quad E_c(t_o) = 28,000 \text{ MPa}; \quad E_p = 195,000 \text{ MPa}; \quad E_c^* = E_c(t^*) = 32,000 \text{ MPa}; \]
\[ A_c = 160,000 \text{ mm}^2; \quad A_p = 1430 \text{ mm}^2 \]

**Time step:** The single time step is infinite and runs from $t_o = 14$ days to $t_1 = t^*$ (time infinity).

**Initial elastic analysis:** The initial prestressing force $P_o$ induces a concrete stress $\sigma_{co}$ and an elastic strain $\varepsilon_{\text{cee}}$:

\[ \sigma_{co} = \frac{P_o}{A_c} = \frac{1600 \times 10^6}{160000} = 10 \text{ MPa} \quad (\text{EQ 4}) \]

\[ \varepsilon_{\text{cee}} = \frac{\sigma_{co}}{E_{co}} = \frac{10}{28000} = 0.00036 \quad (\text{EQ 5}) \]

We also have $\sigma_{po} = 1119$ MPa.

**Free creep in the time step:** With the concrete uncoupled from the tendon, creep occurs freely and the total creep at $t^*$, shown in Figure 2, is $\Delta e_{\text{cc}}$ which for a one-step analysis is equal to $\varepsilon_{\text{cc}}^*$:

\[ \varepsilon_{\text{cc}}^* = \varphi(t^*, t_o)\varepsilon_{\text{cee}} = 3.0 \times 0.00036 = 0.00108 \quad (\text{EQ 6}) \]

**Restoration of compatibility at $t^*$:** To restore compatibility we apply a tensile force $\Delta X_c$ to the concrete and an equal compressive force $\Delta X_p$ to the tendon. These induce elastic strain increments of $\Delta e_{\text{cc}}$ tension in the concrete and $\Delta e_{\text{p}}$ compression in the tendon. As can be seen in Figure 2, strain compatibility is restored if

\[ \Delta e_{\text{cc}} + \Delta e_{\text{p}} = \varepsilon_{\text{cc}}^* \quad (\text{EQ 7}) \]

Using the instantaneous stiffness properties of the materials we have

\[ \Delta e_{\text{cc}} = \frac{\Delta X_c}{(A_c E_c^*)} \quad (\text{EQ 8}) \]
\[ \Delta e_{\text{p}} = \frac{\Delta X_p}{(A_p E_p)} \quad (\text{EQ 9}) \]
where \( E_c^* \) is the elastic concrete modulus at time \( t^* \). Substituting in Equation 7 we obtain the following expressions for the strain increment \( \Delta e_{cc} \) and the force increments:

\[
\Delta e_{cc} = \frac{A_p E_p}{A_p E_p + A_c E_c^*} e_{cc}^*
\]

\[ (EQ \text{10}) \]

\[
\Delta X_c = \Delta X_p = \frac{A_p E_p}{A_p E_p + A_c E_c^*} e_{cc}^* A_c E_c^*
\]

\[ (EQ \text{11}) \]

Substituting values, we have

\[
A_p E_p = 1430 \times 195000 = 0.279 \times 10^9 \text{ N}
\]

\[
A_c E_c^* = 160000 \times 32000 = 5.12 \times 10^9 \text{ N}
\]

\[
\Delta e_{cc} = \frac{0.279}{5.12 + 0.279} \times 0.00108 = 0.0000558
\]

\[
\Delta X_p = \frac{0.279}{5.12 + 0.279} 0.00108 \times 5.12 \times 10^9 = 286 \text{ kN}
\]

The prestressing force at the end of the process at time \( t^* \) is

\[
P^* = P_o - \Delta X_p = 1600 - 286 = 1317 \text{ kN}
\]

\[ (EQ \text{12}) \]

The loss of prestress due to creep is

\[
\Delta X/P_o = 286/1600 = 0.179, \text{ or 17.9 per cent.}
\]

In this case, the per cent loss of prestress in the concrete is of course the same as in the tendon.

**EXAMPLE 2: THREE-STEP ANALYSIS, SYMMETRIC SECTION, NO REINFORCEMENT**

We use the same data for the three-step analysis as for Example 1. The three time steps are shown qualitatively in Figure 3, together with the corresponding step decrements in force in the concrete, the creep strains in the concrete and the relevant creep functions.

**Choice of time steps and additional numerical data**

The three step-changes in stress occur at times \( t_1, t_2 \) and, after the creep process is completed, at \( t^* \). We choose times \( t_1 \) and \( t_2 \) such that the amount of creep that occurs in the three time intervals, \((t_0 \text{ to } t_1), (t_1 \text{ to } t_2), \text{ and } (t_2 \text{ to } t^*)), is about the same. With \( \varphi(t^* \times 14) = 3.0 \) as before, the times \( t_1 \) and \( t_2 \) are chosen such that \( \varphi(t_1, t_{14}) = 1.0 \) and \( \varphi(t_2, t_{14}) = 2.0 \). For first loading at 14 days, and using data from AS 3600-2009 for a theoretical thickness of \( t_h = 200 \text{ mm} \), it is found that equal creep increments occur with times after first loading of approximately 40 and 200 days. To obtain
the corresponding ages of the concrete we need to add 14 days to these values, and thus obtain: \( t_1 = 54 \) days, and \( t_2 = 214 \) days.

In subsequent calculations we will require incremental values of the creep functions \( \varphi(t, t_0) \), \( \varphi(t, t_1) \) and \( \varphi(t, t_2) \), and it is convenient to evaluate them now. Using creep data from AS 3600-2009 we obtain the following values:

\[
\begin{align*}
\varphi(t^*, 14) &= 3.0; \quad \varphi(214, 14) = 2.0; \quad \varphi(54, 14) = 1.0; \\
\varphi(t^*, 54) &= 2.0; \quad \varphi(214, 54) = 1.8; \\
\varphi(t^*, 214) &= 1.9;
\end{align*}
\]

The elastic moduli of the concrete at the various times are taken to be (in MPa): \( E_{co} = 28,000; E_{c1} = 30,000; E_{c2} = 31,000; \) and \( E_{c}^{*} = 32,000. \)

**Initial elastic analysis**

As in Example 1 we have:

\[
P_o = 1,600 \text{ kN}; \quad \sigma_{co} = 10 \text{ MPa}; \quad \varepsilon_{cco} = 0.00036.
\]

The initial stress and strain in the tendon are:

\[
\sigma_{po} = \frac{P_o}{A_p} = 1119 \text{ MPa}; \quad \varepsilon_{po} = \frac{1119}{195 \times 10^3} = 0.0057.
\]

**Step 1 analysis: free creep between 14 and 54 days**

For the load \( P_o \) applied at 14 days, the elastic strain is \( \varepsilon_{cco} = 0.00036. \) The free creep strain increment in this first step is \( \varepsilon_{cc} (54, 14) \), i.e. the creep strain at 54 days due to the constant stress applied at 14 days. We have:

\[
\varepsilon_{cc} (54, 14) = \varepsilon_{cco} \times \varphi(54, 14) = 0.00036 \times 1.0 = 0.00036
\]

This is the total free creep strain increment \( \Delta \varepsilon_{cel} \) in the first time step, and it is also the strain incompatibility between the concrete and the tendon at 54 days.

It is convenient here to calculate the creep strain increments due to \( \sigma_{co} \) that will be needed later in the second and third time step calculations. We have

\[
\varepsilon_{cc} (214, 14) = \varepsilon_{cco} \times \varphi(214, 14) = 0.00036 \times 2.0 = 0.00072;
\]

\[
\varepsilon_{cc} (t^*, 14) = \varepsilon_{cco} \times \varphi(t^*, 14) = 0.00036 \times 3.0 = 0.00108
\]

**Step 1 analysis: restoration of compatibility**

To complete step 1 we restore compatibility at \( t_1 = 54 \) days using a compressive force \( \Delta X_{p1} \) in the tendon and an equal tensile force \( \Delta X_{c1} \) in the concrete. It will be convenient to refer to all such changes in values as "increments", whether or not the value of the quantity is increased or decreased; however, to ensure clarity, the change will be denoted as tensile or compressive. The elastic tensile strain increment induced in the concrete is \( \Delta \varepsilon_{cel} \) and the compressive strain incre-
ment in the tendon is \( \Delta \varepsilon_p1 \). These together must add up to the value \( \Delta \varepsilon_{cc1} \):
\[
\Delta \varepsilon_{cc1} = \Delta \varepsilon_{ce1} + \Delta \varepsilon_{p1}.
\]

With \( \Delta \varepsilon_{ce1} = \Delta X_{c1}/(E_{c1}A_c) \) and \( \Delta \varepsilon_{p1} = \Delta X_{p1}/(E_pA_p) \) we obtain:
\[
\Delta X_{c1} = \Delta \varepsilon_{cc1} E_p A_p/(1 + n_1 p_p), \text{ where } n_1 = E_p/E_{c1}, \text{ and } E_{c1} \text{ is the elastic modulus of the concrete at 54 days. Substituting values we obtain } \Delta X_{c1} = 94.9 \text{ kN. The elastic stress and strain increments for the concrete at 54 days are tensile and have the values } \Delta \sigma_{c1} = 0.6 \text{ MPa, and } \Delta \varepsilon_{cc1} = 0.000020.
\]

**Step 2 analysis: free creep between times 54 and 214 days**

During step 2, free creep of the concrete due to the two sustained stresses \( \sigma_{co} \) and \( \Delta \sigma_{c1} \) are calculated separately. The free compressive creep in step 2 caused by \( \sigma_{co} \) is
\[
\varepsilon_{cc}(214,14) - \varepsilon_{cc}(54,14) = 0.00072 - 0.00036 = 0.00036
\]

Due to \( \Delta \sigma_{c1} \), the free tensile creep at 214 days is
\[
\varepsilon_{cc}(214, 54) = \Delta \varepsilon_{ce1} q(214, 54) = 0.00002 \times 1.8 = 0.000036
\]

The total free compressive creep strain during step 2 is thus
\[
\Delta \varepsilon_{cc2} = 0.00036 - 0.000036 = 0.00032
\]

We will also calculate here the value of \( \varepsilon_{cc}(t^*, 54) \) as it will be needed in the step 3 calculation. With \( q(t^*, 54) = 2.0 \) we have
\[
\varepsilon_{cc}(t^*, 54) = 0.000020 \times 2.0 = 0.00004
\]

**Step 2 analysis: restoration of compatibility**

To complete the step 2 analysis we apply a tensile force \( \Delta X_{c2} \) at 214 days to the concrete and an equal compressive force to the tendon to restore compatibility. The elastic strain increment induced in the concrete by \( \Delta X_{c2} \) is \( \Delta \varepsilon_{ce2} = \Delta X_{c2}/A_c E_{c2} \) and with \( \Delta X_{c2} = \varepsilon_{cc2} E_p A_p/(1 + n_2 p_p) \) we obtain \( \Delta X_{c2} = 84.5 \text{ kN. The tensile strain and stress increments in the concrete are:} \)
\[
\Delta \varepsilon_{ce2} = 0.000017, \text{ and } \Delta \sigma_{c2} = 0.5 \text{ MPa.}
\]

**Step 3 analysis: free creep between 214 days and \( t^* \)**

During step 3 free creep occurs in the concrete under the compressive stress \( \sigma_{co} \) and the two tensile increments \( \Delta \sigma_{c1} \) and \( \Delta \sigma_{c2} \). Due to \( \sigma_{co} \) the free compressive creep is
\[
\varepsilon_{cc}(t^*, 14) - \varepsilon_{cc}(214, 14) = \varepsilon_{cc0}(\varphi(t^*, 14) - \varphi(214, 14)) = 0.00036(3.0 - 2.0)
\]

The value is 0.00036.

The free tensile creep due to \( \Delta \sigma_{c1} \) in this time interval is
\[ \varepsilon_{cc}(t^*, 54) - \varepsilon_{cc}(214, 54) \]

In the previous step 2 analysis we obtained \( \varepsilon_{cc}(214, 54) = 0.000036 \) and \( \varepsilon_{cc}(t^*, 54) = 0.000040 \), which gives the value 0.000004 as the creep strain increment due to \( \Delta \sigma_{c1} \).

Due to \( \Delta \sigma_{c2} \) the free tensile creep is \( \varepsilon_{cc}(t^*, 214) = \Delta \varepsilon_{cc2} \varphi(t^*, 214) \). From the creep data we have \( \varphi(t^*, 214) = 1.9 \) and with \( \Delta \varepsilon_{cc2} = 0.000017 \) we obtain \( \varepsilon_{cc}(t^*, 214) = 0.000032 \).

The compressive sum of the creep strain increments is:

\[ \Delta \varepsilon_{cc3} = 0.00036 - 0.000004 - 0.000032 = 0.00032 \]

**Step 3 analysis: restoration of compatibility**

To restore compatibility we require \( \Delta \varepsilon_{cc3} = \Delta \varepsilon_{cc3} + \Delta \varepsilon_{p3} \)

where \( \Delta \varepsilon_{cc3} = \Delta X_{c3}/(A_p E_p) \) and \( \Delta \varepsilon_{p3} = \Delta X_{p3}/(A_p E_p) \). The final tensile force increment needed to be applied at time \( t^* \) to the concrete to restore compatibility at the end of the last step, is

\[ \Delta X_{c3} = \varepsilon_{cc3} E_p A_p / (1 + n_3 P_p) \]

and substituting values we obtain \( \Delta X_{c3} = 84.6 \) kN.

**Final conditions at end of process**

The total loss in prestress is \( \Delta X_{p1} + \Delta X_{p2} + \Delta X_{p3} = 94.9 + 84.5 + 84.6 = 264 \) kN. The final prestressing force is \( P^* = 1,600 - 264 = 1,336 \) kN, so that the per cent loss is \( (264/1600 \times 10^3) \times 100 = 16.5 \) per cent.

**Discussion**

As one would expect, a three-step analysis gives a smaller value for the prestress loss than the one-step analysis. In this case the difference is small: 17.9 versus 16.5 per cent. Although the accuracy of a three-step analysis will be better than for one step, the amount of calculation has increased markedly. It seems that little would be gained by going to a yet larger number of steps. Indeed, a one-step analysis usually gives acceptable results, if on the conservative side, and we will use one-step analyses from now on in the numerical examples.

4. **Creep in a symmetric prestressed section with reinforcement**

In a symmetric prestressed section containing reinforcement, the compressive creep strains that occur in any time step induce a compressive force increment \( \Delta X_s \) in the steel, as well as the compressive increment \( \Delta X_p \) in the tendon. For equilibrium at the end of a time step, these compressive force increments are equal to the tensile force increment in the concrete:

\[ \Delta X_c = \Delta X_p + \Delta X_s \]  \hspace{1cm} (EQ 13)

An example of a one-step analysis is given in the following example. A multi-step analysis would follow the procedure used in Example 2.
EXAMPLE 3: CREEP IN AN AXIALLY PRESTRESSED MEMBER WITH SYMMETRICAL REINFORCEMENT

For this example we use the same cross-section as for Examples 1 and 2, but with symmetrically placed reinforcement as in Figure 1(b). The area \( A_s = 800 \text{ mm}^2 \), and \( E_s = 200,000 \text{ MPa} \). As before, the post-tensioning tendon has an area of 1430 mm² and the other data are unchanged:

\[
t_0 = 14 \text{ days}; \\
\varphi^*_o = \varphi(t^*, 14) = 3.0; \\
E_{co} = E_c(t_0) = 28,000 \text{ MPa}; \\
E^*_c = E_c(t^*) = 32000 \text{ MPa}; \\
E_p = 195,000 \text{ MPa}; \\
A_c = 160,000 \text{ mm}^2
\]

The initial prestress in the tendon, immediately after the prestressing operation, is \( P_{po} \) and the initial forces in the concrete and steel are \( P_{co} \) and \( P_{so} \). As before, \( P_{po} = 1,600 \text{ kN} \), but \( P_{co} \) is now less than this.

Preliminary elastic analysis:

At time \( t_0 \) the stress and strain in the tendon are \( \sigma_{po} = \frac{P_{po}}{A_p} = 1119 \text{ MPa} \) and \( \varepsilon_{po} = \frac{\sigma_{po}}{E_p} = 0.0057 \). The sum of the compressive forces in the concrete and in the reinforcing steel is equal to the tensile tendon force:

\[
P_{po} = P_{co} + P_{so} \tag{EQ 14}
\]

The initial compressive elastic strain in the concrete is \( \varepsilon_{cco} \), which is also the initial strain in the steel. Expressing \( P_{co} \) and \( P_{so} \) in terms of \( \varepsilon_{cco} \) and substituting into Equation 14 we obtain the following expression:

\[
\varepsilon_{cco} = \frac{P_{po}}{A_s E_s + A_c E_{co}} \tag{EQ 15}
\]

The forces can be expressed as proportions of \( P_{po} \) and also in terms of \( \varepsilon_{cco} \) as follows:

\[
P_{co} = \frac{A_c E_{co}}{A_c E_{co} + A_s E_s} P_{po} = A_c E_{co} \varepsilon_{cco} \tag{EQ 16}
\]

\[
P_{so} = \frac{A_s E_s}{A_c E_{co} + A_s E_s} P_{po} = A_s E_s \varepsilon_{cco} \tag{EQ 17}
\]

We obtain the following numerical values:

\[
P_{co} = \frac{160000 \times 28000}{160000 \times 28000 + 800 \times 200000} \times 1600 = 1545 \text{ kN}
\]

\[
P_{so} = \frac{800 \times 200000}{160000 \times 28000 + 800 \times 200000} \times 1600 = 55 \text{ kN}
\]

July 3, 2012
The initial concrete stress and elastic strain are $\sigma_{c0} = 9.66$ MPa and $\varepsilon_{c0} = 0.000345$. At time $t_0$ the steel is carrying only 55 kN, a very small proportion of the total compressive prestressing force.

**Single time step: free creep**

During the time step from $t_0$ to $t^*$ the free creep strain is

$$
\varepsilon_{cc}^* = \varphi(t^*, t_0)\varepsilon_{c0} \quad \text{(EQ 18)}
$$

which gives

$$
\varepsilon_{cc}^* = \varphi(t^*, t_0)\varepsilon_{c0} = 3.0 \times 0.000345 = 0.001035.
$$

**Single time step: restoring compatibility at $t^*$**

To restore compatibility a tensile force increment $\Delta X_c$ is applied to the concrete, while equilibrating compressive force increments $\Delta X_p$ and $\Delta X_s$ are applied to the tendon and steel, respectively, with:

$$
\Delta X_c = \Delta X_p + \Delta X_s \quad \text{(EQ 19)}
$$

The resulting increment in elastic tensile strain in the concrete is $\Delta \varepsilon_{cc}$. The compressive strain increment in the tendon is $\Delta \varepsilon_p$. The strain increment in the steel, $\Delta \varepsilon_s$, is equal to $\Delta \varepsilon_p$. For compatibility we have:

$$
\Delta \varepsilon_{cc} + \Delta \varepsilon_p = \varepsilon_{cc}^* \quad \text{(EQ 20)}
$$

The tensile force increment in the concrete is

$$
\Delta X_c = A_c E_c^* (\Delta \varepsilon_{cc}) \quad \text{(EQ 21)}
$$

The compressive force increments in the tendon and steel are

$$
\Delta X_p = A_p E_p (\varepsilon_{cc}^* - \Delta \varepsilon_{cc}) \quad \text{(EQ 22)}
$$

$$
\Delta X_s = A_s E_s (\varepsilon_{cc}^* - \Delta \varepsilon_{cc}) \quad \text{(EQ 23)}
$$

Substituting these expressions into the equilibrium requirement, Equation 19, and solving for $\Delta \varepsilon_{cc}$, we obtain

$$
\Delta \varepsilon_{cc} = \frac{E_p A_p + E_s A_s}{E_p A_p + E_s A_s + E_c^* A_c} \varepsilon_{cc}^* \quad \text{(EQ 24)}
$$

To determine the prestress loss in the concrete, $\Delta X_c$, the tensile strain increment $\Delta \varepsilon_{cc}$ is first evaluated from Equation 24 and substituted into Equation 21.

From Equation 24 we obtain:
\[ \Delta e_{cc} = \frac{195000 \times 1430 + 200000 \times 800}{195000 \times 1430 + 200000 \times 800 + 32000 \times 160000} \times 0.001035 \]

which gives the value \( 0.079 \times 0.001035 = 8.15 \times 10^{-5} \)

The force increments are then
\[ \Delta Y_c = 418 \text{ kN}; \; \Delta Y_s = 152 \text{ kN}; \; \Delta Y_p = 266 \text{ kN}. \]

As an equilibrium check, we note that 152 + 266 is equal to 418.

**Loss of prestress**

The per cent changes in the forces are as follows
\[ \Delta Y_c / P_{co} = -27 \%; \; \Delta Y_s / P_{so} = +276 \%; \; \Delta Y_p / P_{po} = -17 \%. \]

**Discussion**

The percentage loss of force in the concrete is now much larger than the loss in the tendon, because there has been a transfer of compressive force from the concrete to the steel reinforcement. In this particular case the calculated loss in force for the tendon is 17 per cent, but is 28 per cent for the concrete. The transfer of compressive force from the concrete to the steel results in a 275 per cent increase in the steel force. In this example the steel proportion is quite small. A larger area of steel would lead to an even larger loss of prestress in the concrete.

5. Creep in members with eccentric prestress and reinforcement

When an eccentric force is applied to a plain concrete cross-section and sustained, the effect of creep is simply to magnify the initial strains in each concrete fibre by a multiple of the elastic strain (Warner et al., 1988). At any depth in the section, the creep strain, \( e_{cc}(t) \), is obtained from the initial elastic strain, \( e_{ce} \), using the creep function:

\[ e_{cc}(t) = \varphi(t, t_o) e_{ce} \quad \text{(Eq 25)} \]

This means that the creep curvature in the section at time \( t \), \( \kappa_c(t) \), is obtained from the initial elastic curvature, \( \kappa_o \), simply by multiplying by the creep coefficient \( \varphi(t, t_o) \):

\[ \kappa_c(t) = \varphi(t, t_o) \kappa_o \quad \text{(Eq 26)} \]

A multiplying factor \( R(t) \) can thus be used to obtain the total strain in a fibre and also the total curvature (elastic plus creep) at time \( t \):

\[ R(t) = [1 + \varphi(t, t_o)] \quad \text{(Eq 27)} \]

The total long-term curvature after all creep has taken place is

\[ \kappa^* = R \kappa_o = (1 + \varphi^*_o) \kappa_o \quad \text{(Eq 28)} \]
and the creep component is

$$\kappa_c^* = \varphi_o^* \kappa_0.$$  \hspace{1cm} (EQ 29)

This means that the deflected shape of a flexural member after all creep has occurred, $y'(x)$, can be obtained from the initial deflection curve using the multiplier $R^*$:

$$y'(x) = R^* y_0(x).$$  \hspace{1cm} (EQ 30)

Clearly the above expressions ignore the presence of the tendon and reinforcing steel in typical cross-sections, and hence do not take account of the progressive transfer of compressive stress from the concrete to the steel and tendon. The progressive loss of prestress in the concrete with time has also been ignored. Equations 25 to 30 can thus lead to significant error when applied to a section with reinforcing steel and tendon. Nevertheless, they can be used to obtain a first approximate estimate of creep strains and curvature in the section of a prestressed beam. A step-by-step analysis can then be used to correct for the presence of the tendon and reinforcing steel.

We will use a one-step analysis to evaluate creep in a rectangular concrete section that contains a tendon of area $A_p$ at eccentricity $e$ below the centroid of the section, and a layer each of tensile and compressive reinforcement, as shown in Figure 1(d). The tensile steel area, $A_{st}$, is at depth $d_s$ and the compressive steel area, $A_{sc}$, is at depth $d_c$. The tendon and the steel are bonded to the surrounding concrete and the initial prestressing force is $P_o$. A sustained dead-load moment $M_G$ is also assumed to act from the time when the prestress is applied. The analysis will be used to obtain a correction factor $\Delta k^*_c$ to be added to the approximate creep curvature $k^*_c$ in Equation 29.

An eccentric prestress produces stress and strain gradients across the section, and the initial elastic concrete strain distribution due to prestress and a sustained moment $M_G$ are shown in Figure 4(b). All concrete strains are assumed to be compressive, with the top fibre concrete strain, $\varepsilon_{cett}$, greater than the bottom fibre strain, $\varepsilon_{csrf}$, so that the initial curvature in the section is positive. This will be the case if $M_G$ is greater than the prestressing moment, $M_p = P_o e$.

To carry out a one-step analysis we allow free creep to take place in the section, ignoring the restraining effect of the bonded tendon and steel. The free long-term creep strains and curvature are calculated as previously using Equations 27 to 31. The free creep strains are shown in Figure 4(c), where $\varepsilon_{cett}$ and $\varepsilon_{cshe}$ are the free creep strain in the top and bottom fibres, respectively. At the level of the tendon the free creep strain in the concrete is $\varepsilon_{cett}^*$. This is also the amount by which the strain in the tendon, after creep, is incompatible with the strain in the adjacent concrete. The value of $\varepsilon_{cett}^*$ is calculated as:

$$\varepsilon_{cett}^* = \varphi_o^* \varepsilon_{cett} = \varphi_o^* \frac{A_c}{E_c} \frac{P_o + (P_o e - M_G)e}{I_c}.$$  \hspace{1cm} (EQ 31)

The term $\varepsilon_{cett}^*$ is the initial elastic concrete strain at the tendon level. A strain incompatibility also exists at each level of reinforcement. The free creep strains in the concrete at the tensile and compressive steel levels, which are also the strain incompatibilities, are $\varepsilon_{cett}^*$ and $\varepsilon_{cscf}^*$, respectively. These strains can be evaluated using equations similar to Equation 31. For example, at the level of the tensile steel, $\varepsilon_{cett}^*$ has the value:
\[
\varepsilon_{c,est}^* = \varepsilon_{c,est}^0 = \sigma_{c,est}^* \frac{P_0 + \frac{(P_0 e - M_G)}{I_c}}{E_c} \left( d_{st} - d_g \right)
\]  
(EQ 32)

The term \(d_g\) is the depth to the centroid of the gross concrete section, which is \(D/2\) in the present case.

To restore compatibility at time \(t^*\) we introduce an equilibrating force system which consists of a tensile force increment in the concrete, \(\Delta P_c\), acting at some depth \(d_{sc}\) below the top fibre, and compressive force increments in the tendon and the steel areas, i.e. \(\Delta P_{st}, \Delta P_{se} \) and \(\Delta P_{sc}\). For longitudinal equilibrium, \(\Delta P_c\) is equal to the sum of the compressive force increments. To satisfy moment equilibrium, the depth \(d_{sc}\) is such that the moment of \(\Delta P_c\) about the top fibre is equal to the sum of the moments of \(\Delta P_{st}, \Delta P_{se}\) and \(\Delta P_{sc}\) about the top fibre.

As \(\Delta P_c\) does not act at the centroid of the section, it produces an elastic strain gradient in the concrete, with tensile strain increments at the top and bottom fibres of \(\Delta e_{c,ea}\) and \(\Delta e_{c,eb}\), respectively. The incremental tensile strain gradient in the concrete is shown in Figure 4(c). At the level of the tendon the tensile strain increment in the concrete is expressed in terms of the extreme fibre strain increments as:

\[
\Delta e_{c,et} = \Delta e_{c,ea} \frac{D - d_g}{D} + \Delta e_{c,eb} \frac{d_g}{D}
\]  
(EQ 33)

At this level the compressive strain increment in the tendon, required to restore compatibility, is seen from Figure 4(d) to be

\[
\Delta e_p = \varepsilon_{c,et}^* - \Delta e_{c,et}.
\]  
(EQ 34)

The increment of compressive force in the tendon is thus

\[
\Delta X_p = A_p E_p (\varepsilon_{c,et}^* - \Delta e_{c,et})
\]  
(EQ 35)

Likewise, for the two steel layers the elastic concrete tensile strain increments are \(\Delta e_{c,est}\) and \(\Delta e_{c,exe}\) which can also be expressed in terms of the extreme fibre strain increments using equations very similar to Equation 33, with only the term \(d_p\) replaced by the depths to the steels, \(d_{st}\) and \(d_{sc}\). For example, for the tensile steel, we have

\[
\Delta e_{c,est} = \Delta e_{c,esa} \frac{D - d_{st}}{D} + \Delta e_{c,ese} \frac{d_{st}}{D}
\]  
(EQ 36)

and the compressive force increment in this steel is

\[
\Delta X_{st} = A_{st} E_p (\varepsilon_{c,est}^* - \Delta e_{c,est}).
\]  
(EQ 37)

Similar equations apply to the second steel level, the only difference being that \(A_{st}\) is replaced by \(A_{sc}\) and \(d_{st}\) is replaced by \(d_{sc}\) in Equations 36 and 37.
For the rectangular section that we are considering, the tensile force increment in the concrete is

$$\Delta X_c = A_e E'_c \left( \frac{\Delta \varepsilon_{cea} + \Delta \varepsilon_{ceb}}{2} \right)$$  \hspace{1cm} (EQ 38)

which acts at depth \(d_{xc}\):

$$d_{xc} = \frac{A_e E'_c \left( \frac{\Delta \varepsilon_{cea} D}{6} + \frac{\Delta \varepsilon_{ceb} D}{3} \right)}{\Delta X_c} = \frac{\frac{1}{3} \Delta \varepsilon_{cea} + \frac{2}{3} \Delta \varepsilon_{ceb}}{\Delta \varepsilon_{cea} + \Delta \varepsilon_{ceb}} D$$  \hspace{1cm} (EQ 39)

The above equations express all unknown quantities in terms of the top and bottom fibre strain increments, \(\Delta \varepsilon_{cea}\) and \(\Delta \varepsilon_{ceb}\), which define the increment in the strain gradient, and hence the correction to the creep curvature,

$$\Delta k^* = \frac{(\Delta \varepsilon_{cea} - \Delta \varepsilon_{ceb})}{D}$$  \hspace{1cm} (EQ 40)

To evaluate these strain increments we apply the equations of force equilibrium and moment equilibrium to the increments in forces:

$$\Delta X_c = \Delta X_p + \Delta X_{st} + \Delta X_{sc}$$  \hspace{1cm} (EQ 41)

$$d_{xc} \Delta X_c = d_p \Delta X_p + d_{st} \Delta X_{st} + d_{sc} \Delta X_{sc}$$  \hspace{1cm} (EQ 42)

The force increments in these equations and \(d_{xc}\) have already been expressed in terms of \(\Delta \varepsilon_{cea}\) and \(\Delta \varepsilon_{ceb}\) and we thus obtain two linear equations in \(\Delta \varepsilon_{cea}\) and \(\Delta \varepsilon_{ceb}\), which can be written as

$$a_{11} \Delta \varepsilon_{cea} + a_{12} \Delta \varepsilon_{ceb} = c_1$$  \hspace{1cm} (EQ 43)

$$a_{21} \Delta \varepsilon_{cea} + a_{22} \Delta \varepsilon_{ceb} = c_2$$  \hspace{1cm} (EQ 44)

or

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \Delta \varepsilon_{cea} \\ \Delta \varepsilon_{ceb} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$  \hspace{1cm} (EQ 45)

The values of \(\Delta \varepsilon_{cea}\) and \(\Delta \varepsilon_{ceb}\) are thus

$$\Delta \varepsilon_{cea} = \frac{a_{22} c_1 - a_{12} c_2}{a_{11} a_{22} - a_{21} a_{12}}$$  \hspace{1cm} (EQ 46)

$$\Delta \varepsilon_{ceb} = \frac{a_{11} c_2 - a_{21} c_1}{a_{11} a_{22} - a_{21} a_{12}}$$  \hspace{1cm} (EQ 47)
For a specific cross-section, the above analysis can of course be carried out numerically from first principles. This is illustrated in Example 4 for the simple case of a rectangular section that contains only the tendon, but no reinforcement.

Alternatively, the $a_{ij}$ parameters in Equation 45 can be evaluated, and used to determine $\Delta \varepsilon_{c_{ea}}$ and $\Delta \varepsilon_{c_{eb}}$ from Equations 46 and 47. The calculations are best undertaken using a spreadsheet. The parameters are as follows:

$$a_{11} = \frac{1}{2} A_c E_c^* + A_p E_p \left( \frac{D - d_p}{D} + A_{s_{st}} E_s \frac{d_{st}}{D} + A_{s_{sc}} E_s \frac{d_{sc}}{D} \right)$$  \hspace{1cm} (EQ 48)

$$a_{12} = \frac{1}{2} A_c E_c^* + A_p E_p \frac{d_p}{D} + A_{s_{st}} E_s \frac{d_{st}}{D} + A_{s_{sc}} E_s \frac{d_{sc}}{D}$$  \hspace{1cm} (EQ 49)

$$c_1 = A_p E_p \varepsilon_{c_{ccp}}^* + A_{s_{st}} E_s \varepsilon_{c_{cost}}^* + A_{s_{sc}} E_s \varepsilon_{c_{ccsc}}^*$$  \hspace{1cm} (EQ 50)

$$a_{21} = A_c E_c^* \frac{D}{6} + A_p E_p \frac{d_p}{D} \frac{D - d_p}{D} + A_{s_{st}} E_s \frac{d_{st}}{D} \frac{D - d_{st}}{D} + A_{s_{sc}} E_s \frac{d_{sc}}{D} \frac{D - d_{sc}}{D}$$  \hspace{1cm} (EQ 51)

$$a_{22} = A_c E_c^* \frac{D}{3} + A_p E_p \frac{d_p}{D} + A_{s_{st}} E_s \frac{d_{st}}{D} + A_{s_{sc}} E_s \frac{d_{sc}}{D}$$  \hspace{1cm} (EQ 52)

$$c_2 = A_p E_p \varepsilon_{c_{ccp}}^* + A_{s_{st}} E_s \varepsilon_{c_{cost}}^* + A_{s_{sc}} E_s \varepsilon_{c_{ccsc}}^*$$  \hspace{1cm} (EQ 53)

If an analysis of a non-rectangular section were undertaken, only the terms $c_1$ and $c_2$ would have to be modified slightly. With the strain increments evaluated, the correction to the curvature can be calculated using Equation 40.

The improved estimate of the creep curvature is then

$$\kappa_c^{**} = \kappa_c^* + \Delta \kappa_c^*$$  \hspace{1cm} (EQ 54)

which can be rewritten as

$$\kappa_c^{**} = \alpha \kappa_c^*$$  \hspace{1cm} (EQ 55)

where the correction factor $\alpha$ is

$$\alpha = 1 + \frac{\Delta \kappa_c^*}{\kappa_c^*}$$  \hspace{1cm} (EQ 56)

and can be smaller or greater than unity, depending on the sign of $\Delta \kappa_c^*$.

A note on signs and sign convention is appropriate here. In the present analyses, compressive concrete strain is taken as positive. This means that the initial elastic curvature due to prestress is negative in the positive moment region of a beam near mid-span if the prestressing tendon is located in
the lower part of the section, with \( d_p > d_{eg} \). The curvature due to the positive sustained moment, \( M_G \), is positive. The free creep curvature \( \kappa_c^* \) has the same sign as the initial curvature, and hence will be positive provided \( M_G \) is greater than \( M_p \).

Irrespective of whether the initial elastic curvature, due to prestress and sustained moment \( M_G \), is positive or negative, the correction to the creep curvature, \( \Delta \kappa_c^* \), will be positive if the equilibrating tensile force increment \( \Delta X_c \) in the concrete acts below the centroid of the section; that is, if \( d_{xc} \) is greater than 0.5\( D \). If this is the case, the tensile strain increment in the lowermost fibre, \( \Delta \varepsilon_{ceb} \), is greater than \( \Delta \varepsilon_{ces} \) in the top fibre. The value of \( d_{xc} \) depends jointly on the locations and magnitudes of the steel areas and tendon area, and also on the original elastic strain distribution in the section. If the tendon and reinforcing steel are located in the lower fibres, then the compressive force increments must also be located there, and hence also the equilibrating tensile force in the concrete, \( \Delta X_c \). When the steel area is distributed throughout the section, and especially when there is a preponderance of compressive steel close to the top surface, then the location of \( \Delta X_c \) will also depend on the strain distribution, and whether the top fibre strains are in tension or compression. This will be seen more clearly in Example 5, where various cross-sections are analysed for creep effects.

**EXAMPLE 4: CREEP IN AN ECCENTRICALLY PRESTRESSED SECTION**

The creep curvature is calculated for a rectangular section, 400 mm by 800 mm, that contains a prestressed tendon with eccentricity \( e = 250 \text{ mm} \) and an initial force of \( P_0 = 1200 \text{ kN} \). The area of the tendon is \( A_p = 1000 \text{ mm}^2 \). A sustained moment of \( M_G = 400 \text{ kNm} \) acts on the section. We take \( \varphi^* = 2.5, E_{co} = 30,000 \text{ MPa and } E_c = 32,000 \text{ MPa} \).

**Initial conditions**

Considering only the prestress (without \( M_G \) acting) the negative elastic curvature is

\[
\kappa_{ep} = \frac{P_0 e}{E_c J_c} = \frac{1200 \times 10^3 \times 250}{30 \times 10^3 \times 17.07 \times 10^9} = -0.586 \times 10^{-6} \text{ mm}^{-1}
\]

Now considering only the sustained load and the positive moment \( M_G \), the elastic curvature is

\[
\kappa_{eG} = \frac{400 \times 10^6}{30 \times 10^3 \times 17.07 \times 10^9} = 0.781 \times 10^{-6} \text{ mm}^{-1}
\]

The net initial curvature, \( \kappa_0 = \kappa_{ep} + \kappa_{eG} \) is positive. Its value is

\[
0.781 \times 10^{-6} - 0.586 \times 10^{-6} = +0.195 \times 10^{-6} \text{ mm}^{-1}
\]

**Approximate calculation of creep curvature \( \kappa_c^* \)**

The uncorrected creep curvature is obtained from Equation 27 as

\[
\kappa_c^* = 2.5 \times 0.195 \times 10^{-6} = 0.488 \times 10^{-6} \text{ mm}^{-1}
\]

Using Equation 31, at the level of the tendon we obtain:
\[ \varepsilon_{\text{cep}} = \frac{1}{30 \times 10^3} \left( \frac{1200 \times 10^3}{320 \times 10^3} + \frac{1200 \times 10^3 \times 250^2}{17.07 \times 10^9} - \frac{400 \times 10^6 \times 250}{17.07 \times 10^9} \right) \]
\[ = 0.0000762, \text{ hence} \]
\[ \varepsilon_{\text{cep}}^* = 2.5 \times 0.0000762 = 0.000190 \]

**Correction: curvature increment \( \Delta \kappa_c^* \)**

To correct for the presence of the tendon, we apply a tensile force \( \Delta X_c \) to the concrete at the tendon level, causing strain \( \Delta \varepsilon_{\text{cep}} \), and a compressive force \( \Delta X_p \) to the tendon, with strain \( \Delta \varepsilon_p \).

The concrete stress and strain at tendon level due to \( \Delta X_c \) are:

\[
\Delta \sigma_{\text{cep}} = \frac{\Delta X_c}{320 \times 10^3} + \frac{\Delta X_c \times 250^2}{17.07 \times 10^9} = 6.787 \times 10^{-6} \Delta X_c
\]
\[
\Delta \varepsilon_{\text{cep}} = \frac{6.787 \times 10^{-6} \Delta X_c}{32 \times 10^3} = 2.121 \times 10^{-10} \Delta X_c
\]

The compressive increment in tendon strain is:

\[
\Delta \varepsilon_p = \frac{\Delta X_p}{1000 \times 200 \times 10^3} = 5.0 \times 10^{-9} \Delta X_p
\]

For compatibility, \( \Delta \varepsilon_{\text{cep}} + \Delta \varepsilon_p = \varepsilon_{\text{cep}} \), giving:

\[
\Delta X_c \left( 2.121 \times 10^{-10} + 5.0 \times 10^{-9} \right) = 0.000190
\]

Solving: \( \Delta X_c = 36,450 \text{ N} \)

The stresses due to \( \Delta X_c \) in the top fibre and bottom fibre are, respectively:

\[
\Delta \sigma_{\text{ces}} = -\frac{36450}{320000} + \frac{36450 \times 250 \times 400}{17.07 \times 10^9} = (-0.114) + 0.214 = 0.100 \text{ MPa}
\]
\[
\Delta \sigma_{\text{ceb}} = (-0.114) - 0.214 = -0.328 \text{ MPa}
\]

The curvature correction due to \( \Delta X_c \) is positive and is obtained from Equation 40:

\[ \Delta \kappa_c^{**} = \frac{(0.328 - (-0.1))}{800 \times 32 \times 10^3} = 0.0167 \times 10^{-6} \text{ mm}^{-1} \]

The corrected creep curvature is:

\[ \kappa_c^{**} = \kappa_c^* - \Delta \kappa_c^* = (0.488 + 0.017) \times 10^{-6} = 0.5 \times 10^{-6} \]
In this example $A_e$ is a very small proportion of $A_s$ and the correction is only 3.5 per cent. The correction can be much larger than this if the tendon area is larger, or if reinforcing steel is present. This will be seen in Example 5 below and in the results summarised in Table 1

**EXAMPLE 5: CURVATURE CORRECTIONS $\Delta \kappa^*_s$ FOR VARIOUS CROSS-SECTIONS**

To examine the size of the curvature corrections that can occur in a typical rectangular cross-section, creep curvatures are calculated for a 400 mm by 800 mm section, with various reinforcement details and $q^* = 2.5$. A spreadsheet was prepared using the equations developed above. Two values of $M_G$ were used: 400 kNm as in Example 4, and 100 kNm, which is smaller than $M_p$. The simplified estimates of creep curvature, $\kappa^*_s$, are independent of the reinforcing details and are $0.488 \times 10^{-5} \text{ mm}^{-1}$ for $M_G = 400$ kNm, and $-0.977 \times 10^{-6} \text{ mm}^{-1}$ for $M_G = 100$ kNm.

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<th>Top Reinf $A_{sc}$</th>
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<td>10</td>
<td>0</td>
<td>1350</td>
<td>400</td>
<td>-0.068</td>
<td>-13.9</td>
</tr>
<tr>
<td>11</td>
<td>2700</td>
<td>2700</td>
<td>400</td>
<td>-0.098</td>
<td>-20.0</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>2700</td>
<td>400</td>
<td>-0.143</td>
<td>-29.2</td>
</tr>
</tbody>
</table>

The results are presented in Table 1. It is clear that a positive curvature correction, $\Delta \kappa^*_s$, occurs if there is a preponderance of bottom reinforcement. This can be seen from Cases 1 to 4. While top reinforcement tends to produce a negative correction, the sign of $\Delta \kappa^*_s$ then depends in part on the magnitude of $M_G$. This can be seen by comparing the results for Cases 6 and 12, where there is only top reinforcement in addition to the tendon, and 5 and 10. For example, in Case 12, with $M_G = 400$ kNm, the initial curvature due to prestress plus $M_G$ is positive and the compressive free creep strain at the level of the top steel is large. The compressive force $\Delta P_{sc}$ is therefore also large,

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211 kN compared with only 40 kN for $\Delta P_p$ in the tendon, and the resultant of these two forces thus lies above the centroidal axis, as must then the equilibrating resultant tensile concrete force, so that the curvature correction is negative.

In Case 6, with $M_G = 100$ kNm, the initial curvature due to prestress plus $M_G$ is negative and the compressive stress at the level of the top steel is very small, as is the resulting free creep strain there. The compressive force $\Delta P_{sc}$ is also small, only 11 kN compared with 107 kN for $\Delta P_p$ in the tendon. The resultant of these two forces is therefore below the centroidal axis and the curvature correction is positive.

For the sections considered here, the correction varies from -30 per cent to +20 percent. This corresponds to a variation in the correction factor $\alpha$ from 0.7 to 1.2. It should be noted that these numerical results are for a relatively modest value of $\phi^*_c$ of only 2.5.


When concrete shrinkage occurs in a prestressed section it is restrained by the tendon and the reinforcing steel in a manner comparable to the restraint provided against creep. In a non-symmetric section, the restraint to shrinkage results in a strain gradient, and hence curvature and deflection. If the steel and tendon are close to the bottom ‘tensile’ face, the shrinkage strain is smaller in the bottom fibre than in the top fibre, so that the shrinkage curvature, $\kappa_{sh}(t)$, will be positive. The value of $\kappa_{sh}(t)$ depends on the geometric properties of the section, the location of the restraining reinforcement and tendon, and the free shrinkage, which is not denoted here as $c^*_c$ but as $c_{cs}$, in order to follow AS 3600 usage. A step-by-step analysis can be used to estimate the shrinkage curvature, and hence deflection, that occurs over time due to shrinkage. We use a one-step-analysis to study shrinkage in rectangular concrete sections similar to those used previously in the creep analyses. The analysis is very similar to that for creep, and some of the equations turn out to be identical.

Considering the cross-section shown in Figure 1(d), we allow the concrete to shrink freely, assuming no bond between the concrete and the steel and tendon. Uniform shortening of the concrete thus occurs with a strain of $c_{cs}$. As no strain increment occurs in the steel or tendon during shrinkage, the strain incompatibility between the concrete and the reinforcing steel and tendon is also $c_{cs}$. To restore compatibility at time $t^*$ we introduce a tensile force increment in the concrete, $\Delta P_c$, and compressive force increments in the tendon, $\Delta P_p$, and the steel layers, $\Delta P_{st}$ and $\Delta P_{sc}$. The force $\Delta P_c$ is eccentric and produces an elastic strain gradient in the concrete, with top and bottom fibre tensile strains of $\Delta \epsilon_{cra}$ and $\Delta \epsilon_{cbs}$. The compressive strain increment in the tendon is $(\epsilon_{cs} - \Delta \epsilon_{ctb})$, where $\Delta \epsilon_{ctb}$ is the concrete strain increment at the level of the prestressing tendon. The strain increments in the steel layers are, similarly, $(\epsilon_{cs} - \Delta \epsilon_{cest})$ and $(\epsilon_{cs} - \Delta \epsilon_{cesl})$.

As in the creep analysis, we write expressions for the strain increments and force increments in terms of the extreme fibre increments $\Delta \epsilon_{cra}$ and $\Delta \epsilon_{ctb}$. We have

$$\Delta P_c = \Delta \epsilon_{cra} \frac{A_c E_c^*}{2} + \Delta \epsilon_{cbs} \frac{A_c E_c^*}{2} \tag{EQ 57}$$

$$\Delta P_p = \left[ \epsilon_{cs} - \frac{(\Delta \epsilon_{cra} (D-d_p)}{D} + \Delta \epsilon_{ctb} \frac{d_p}{D} \right] A_p F_p \tag{EQ 58}$$

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\[ \Delta P_{st} = \left[ \varepsilon_{cs} - \left( \Delta \varepsilon_{cea} \frac{(D - d_{st})}{D} + \Delta \varepsilon_{ceb} \frac{d_{st}}{D} \right) \right] A_{st} E_s \]  
(EQ 59)

\[ \Delta P_{sc} = \left[ \varepsilon_{cs} - \left( \Delta \varepsilon_{cea} \frac{(D - d_{sc})}{D} + \Delta \varepsilon_{ceb} \frac{d_{sc}}{D} \right) \right] A_{sc} E_s \]  
(EQ 60)

For force equilibrium
\[ \Delta P_c = \Delta P_p + \Delta P_{st} + \Delta P_{sc} \]  
(EQ 61)

Substituting in Equation 61 and rearranging, we obtain a linear equation in \( \Delta \varepsilon_{cea} \) and \( \Delta \varepsilon_{ceb} \):
\[ a_{11}(\Delta \varepsilon_{cea}) + a_{12}(\Delta \varepsilon_{ceb}) = c_1 \]  
(EQ 62)

A second equilibrium requirement is that the moment of the tensile force increment \( \Delta P_c \) about the top fibre of the section is equal to the moments of the force increments of the tendon and steel about the same fibre. The moment of \( \Delta P_c \) is
\[ d_{xe} \Delta P_c = \left[ \frac{1}{6} \Delta \varepsilon_{cea} + \frac{1}{3} \Delta \varepsilon_{ceb} \right] DA_c E_c^* \]  
(EQ 63)

The moment equilibrium requirement leads to a second equation
\[ a_{21}(\Delta \varepsilon_{cea}) + a_{22}(\Delta \varepsilon_{ceb}) = c_2 \]  
(EQ 64)

As with the creep analysis we have
\[ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \Delta \varepsilon_{cea} \\ \Delta \varepsilon_{ceb} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \]  
(EQ 65)

The values of the strain increments are as follows:
\[ \Delta \varepsilon_{cea} = \frac{a_{22} c_1 - a_{12} c_2}{a_{11} a_{22} - a_{21} a_{12}} \]  
(EQ 66)
\[ \Delta \varepsilon_{ceb} = \frac{a_{11} c_2 - a_{21} c_1}{a_{11} a_{22} - a_{21} a_{12}} \]  
(EQ 67)

In this shrinkage analysis, the parameters in the above equations have the following values:
\[ a_{11} = \frac{1}{2} A_c E_c^* + A_p E_p \frac{D - d_p}{D} + A_{st} E_s \frac{D - d_{st}}{D} + A_{sc} E_s \frac{D - d_{sc}}{D} \]  
(EQ 68)
\[ a_{12} = \frac{1}{2} A_c E_c^* + A_p E_p \frac{d_p}{D} + A_{st} E_s \frac{d_{st}}{D} + A_{sc} E_s \frac{d_{sc}}{D} \]  
(EQ 69)
\[ c_1 = E_{cs}(A_p E_p + A_{st} E_s + A_{sc} E_s) \]  
(EQ 70)

\[ a_{21} = \frac{1}{2} A_c E_c^* D \frac{D}{3} + A_p E_p \frac{D - d_p}{D} dp + A_{st} E_s \frac{D - d_{st}}{D} d_{st} + A_{sc} E_s \frac{D - d_{sc}}{D} d_{sc} \]  
(EQ 71)

\[ a_{22} = A_c E_c^* D + A_p E_p \frac{d_p}{D} dp + A_{st} E_s \frac{d_{st}}{D} d_{st} + A_{sc} E_s \frac{d_{sc}}{D} d_{sc} \]  
(EQ 72)

\[ c_2 = E_{cs}(A_p E_p \frac{d_p}{D} + A_{st} E_s d_{st} + A_{sc} E_s d_{sc}) \]  
(EQ 73)

By first evaluating the tensile strain increments in the top and bottom fibres, \( \Delta e_{cea} \) and \( \Delta e_{ceb} \), we determine the shrinkage curvature:

\[ \kappa_s = \frac{\Delta e_{ceb} - \Delta e_{cea}}{D} \]  
(EQ 74)

To calculate the shrinkage deflection for a complete beam it is first necessary to evaluate \( \kappa_{sh} \) at several critical cross-sections along the beam, and then either to use a simplified expression for deflection, or to undertake a numerical integration of shrinkage curvatures.

**EXAMPLE 6: SHRINKAGE WARPING IN A PRESTRESSED SECTION WITH REINFORCEMENT**

We calculate the shrinkage warping in several sections similar to those used in Example 5, with varying amounts of reinforcement. A final shrinkage strain of \( e_{cs} = 0.0006 \) is assumed. For all cases we take \( A_p = 1000 \text{ mm}^2 \) and \( P_0 = 1200 \text{ kN} \).

**Case 1: \( A_{st} = A_{sc} = 0 \)**

For this simple case we work from first principles rather than use the equations developed above.

We allow the concrete to shrink freely, so that \( e_{cs} = 0.0006 \), and then, to restore compatibility, we apply a tensile force \( \Delta P \) to the concrete at the tendon location, and an equal compressive force \( \Delta P \) to the tendon.

At any depth \( d \), the elastic tensile strain in the concrete due to \( \Delta P \) is:

\[ \Delta e_{cxy} = \frac{\Delta P}{32 \times 10^3} \times \left( \frac{1}{320 \times 10^3} - \frac{250 \times (400 - d)}{170.7 \times 10^9} \right) \]

\[ = \Delta P(4.577 \times 10^{-13} \times d - 8.541 \times 10^{-11}) \]

At the top fibre, \( d = 0 \): \( \Delta e_{cea} = -8.541 \times 10^{-11} \Delta P \)

At the bottom fibre, \( d = 800 \): \( \Delta e_{ceb} = 2.807 \times 10^{-10} \Delta P \)

At the tendon level, \( d = 650 \): \( \Delta e_{cep} = 2.121 \times 10^{-10} \Delta P \)

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The elastic strain in the tendon due to $\Delta P$ is:

$$\Delta \varepsilon_p = \frac{\Delta P}{1000 \times 195000} = 5.128 \times 10^{-9} \Delta P$$

For compatibility, $\Delta \varepsilon_{cep} + \Delta \varepsilon_p = \varepsilon_{cs}$, so that

$$\Delta P(2.121 \times 10^{-10} + 5.128 \times 10^{-9}) = 0.0006 \text{ and } \Delta P = 112,350 \text{ N}$$

The extreme fibre strains are:

$$\Delta \varepsilon_{cea} = -8.541 \times 10^{-11} \times 112350 = -9.601 \times 10^{-6}$$

$$\Delta \varepsilon_{ceb} = 2.807 \times 10^{-10} \times 112350 = 31.544 \times 10^{-6}$$

Hence the curvature due to shrinkage is:

$$\kappa_{sh} = \frac{31.544 + 9.601}{800} \times 10^{-6} = 0.051 \times 10^{-6} \text{ mm}^{-1}$$

The loss in prestressing force due to shrinkage is $112.5/1200 = 0.094$ or 9.4 per cent.

**Case 2: $A_{st} = 1350 \text{ mm}^2$, $A_{sc} = 0$**

Substituting in Equations 70 to B.75 with $A_c = 320,000 \text{ mm}^2$, $A_p = 1000 \text{ mm}^2$, $A_{st} = 1350 \text{ mm}^2$, $A_{sc} = 0$, $D = 800 \text{ mm}$, $d_p = 650 \text{ mm}$, $d_{st} = 700 \text{ mm}$, $E_c = 32,000 \text{ MPa}$, $E_p = 195,000 \text{ MPa}$, $E_s = 200,000 \text{ MPa}$ gives the following values for the coefficients:

$$a_{11} = 5.190 \times 10^9 \quad a_{12} = 5.515 \times 10^9 \quad c_1 = 2.790 \times 10^5$$

$$a_{21} = 1.413 \times 10^{12} \quad a_{22} = 2.999 \times 10^{12} \quad c_2 = 1.895 \times 10^8$$

From Equations 66 and 67 the extreme fibre strains are:

$$\Delta \varepsilon_{cea} = -2.676 \times 10^{-5} \quad \Delta \varepsilon_{ceb} = 7.577 \times 10^{-5}$$

and from Equation 74, the shrinkage curvature is:

$$\kappa_s = 0.128 \times 10^{-6} \text{ mm}^{-1}.$$ 

The presence of the tensile reinforcing steel has more than doubled the curvature due to shrinkage warping.

**Other cases:**

Four additional cases have been calculated, and the results for all six cases are summarised in Table 2. The tendon area is constant, $A_p = 1000 \text{ mm}^2$. The tabulated results show clearly that, as would be expected, tensile reinforcement tends to produce positive shrinkage curvature, while compressive reinforcement tends to produce negative curvature due to shrinkage. The values of $\kappa_{sh}$ in Table 2 are comparable in magnitude to values of $\Delta \varepsilon_c$ in Table 1. Shrinkage warping can be large, and have a very significant effect on deflections, in slender shallow members.
TABLE 2.

<table>
<thead>
<tr>
<th>Case</th>
<th>$A_{st}$ mm$^2$</th>
<th>$A_{sc}$ mm$^2$</th>
<th>Shrinkage curvature mm$^{-1}$ x 10$^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>0</td>
<td>0</td>
<td>+ 0.196</td>
</tr>
<tr>
<td>(2)</td>
<td>1350</td>
<td>0</td>
<td>+ 0.128</td>
</tr>
<tr>
<td>(3)</td>
<td>2700</td>
<td>0</td>
<td>+ 0.051</td>
</tr>
<tr>
<td>(4)</td>
<td>1350</td>
<td>1350</td>
<td>+ 0.018</td>
</tr>
<tr>
<td>(5)</td>
<td>0</td>
<td>1350</td>
<td>- 0.059</td>
</tr>
<tr>
<td>(6)</td>
<td>0</td>
<td>2700</td>
<td>- 0.158</td>
</tr>
</tbody>
</table>

7. Simplified expression for creep curvature correction, $\Delta \kappa_c$

Simple, closed-form expressions can be derived for the creep correction curvature $\Delta \kappa_c$, and also for the shrinkage warping curvature $\kappa_{sb}$, if several simplifying assumptions are introduced. These expressions are much easier to use in design calculations than a numerical step-by-step analysis. However, the assumptions limit the applicability of the expressions.

If we ignore for the moment the presence of any reinforcing steel in the section, Equation 41 reduces to the requirement $\Delta X_c = \Delta X_p$. Equations 48 to 53 simplify accordingly and a closed form expression for $\Delta X_c$ can be obtained as follows:

$$\Delta X_c = \gamma_1 A_c E_c^* e_{cep}$$  \hspace{1cm} (EQ 75)

The term $\gamma_1$ is a function of the cross-section properties:

$$\gamma_1 = \frac{1}{1 + \frac{\varepsilon^* A_c}{\varepsilon^* p_p I_c}}$$  \hspace{1cm} (EQ 76)

where the modular ratio at $t'$ is $\varepsilon^* p_p = E_p^*/E_c^*$ and the proportion of tendon area is $p_p = A_p/A_c$.

The tensile strain increments in the concrete at the top and bottom fibres due to $\Delta X_c$ are $\Delta e_{cea}$ and $\Delta e_{ceb}$. Expressed in terms of $\Delta X_c$, these elastic strain increments at the extreme fibres are:

$$\Delta e_{cea} = \frac{\Delta X_c}{A_g E_c^*} - \frac{\Delta X_c e_d b}{I_c E_c^*}$$  \hspace{1cm} (EQ 77)

$$\Delta e_{ceb} = \frac{\Delta X_c}{A_g E_c^*} + \frac{\Delta X_c e_d b}{I_c E_c^*}$$  \hspace{1cm} (EQ 78)
The terms $d_a$ and $d_b$ are the distances to the top and bottom fibres from the centroid of the concrete section. The total strain increments at the extreme fibres due to free creep and $\Delta K_c$, with compression taken as positive, are:

$$\varepsilon_{cba}^* = \varepsilon_{cca} - \Delta \varepsilon_{cea}$$

(EQ 79)

$$\varepsilon_{cbb}^* = \varepsilon_{cbb} - \Delta \varepsilon_{cbb}$$

(EQ 80)

The overall curvature due to creep, corrected for the presence of the prestressing steel, is

$$k_c** = (\varepsilon_{cba}^* - \varepsilon_{cbb}^*) / D$$

(EQ 81)

The creep curvature correction is obtained from the elastic tensile increments:

$$\Delta k_c^* = (\Delta \varepsilon_{cbb} - \Delta \varepsilon_{cca}) / D$$

(EQ 82)

Substituting terms and rearranging we obtain

$$\Delta k_c^* = \gamma_1 \gamma_2 (\varepsilon_{cbb}^*/D)$$

(EQ 83)

In this expression $\varepsilon_{cbb}^*$ is the final free creep strain in the concrete at the steel level, as obtained from the approximate analysis ignoring the presence of the tendon, and given previously by Equation 31. The non-dimensional parameter $\gamma_2$ is a function of the section properties:

$$\gamma_2 = \frac{A_v e D}{I_c}$$

(EQ 84)

Provided the tendon is located below the centroid of the concrete, $\Delta k_c^*$ is positive. To evaluate $\Delta k_c^*$ we first determine the elastic strain $\varepsilon_{cbb}^*$ in the concrete at the level of the tendon when the section is subjected to prestress plus the sustained moment $M_c$, and then calculate the free creep at this level, $\varepsilon_{cbb}^*$, using Equation 31. Finally, we evaluate $\gamma_1$ and $\gamma_2$ and hence $\Delta k_c^*$ using Equations 76, 84 and then 83.

An approximation is now introduced to allow for tensile reinforcement in the section. Provided the tensile steel and the prestressing tendon are reasonably close together in the section, we can replace the area $A_p$ at depth $d_p$ by an equivalent combined steel-tendon area,

$$A_{eq} = A_p + A_{st}$$

(EQ 85)

at an equivalent depth

$$d_{eq} = \frac{A_p d_p + A_{st} d_{st}}{A_{eq}}$$

(EQ 86)

We also take the elastic moduli of the two materials to be equal, with value $E_p$. The eccentricity $e$ is replaced by $e_{eq} = d_{eq} - d_c$, where $d_c$ is the depth of the centroid of the concrete section. These terms can now be used in Equations 76, 84 and 83 to determine the creep curvature correction, $\Delta k_c^*$. This approximation can not be used for compressive reinforcement because of the assumption that the steel and the tendon are physically close in the section.
The added tensile reinforcement in the bottom fibres of the section reduces the compressive creep strains, but provides little restraint to creep in the top fibres, so that the correction is positive, irrespective of whether the initial elastic curvature and the free creep curvature are positive or negative.

EXAMPLE 7: SIMPLIFIED CREEP CURVATURE CALCULATION

The simplified closed form expression (Equation 84) is used to re-calculate $\Delta \kappa_c^*$ for the cross-sections listed as Cases 1, 2, 3 and 4 in Example 6. The following data apply:

\[ A_c = 320,000 \text{ mm}^2; \quad A_p = 1000 \text{ mm}^2; \quad I_c = 17.07 \times 10^9 \text{ mm}^4; \]

\[ D = 800 \text{ mm}; \quad E_c^* = 32,000 \text{ MPa}; \quad e = 250 \text{ mm}; \]

The areas $A_{st}$ for the four cases are, respectively, 2700, 1350, 0 and 1350 mm$^2$. The area of compressive steel $A_{sc}$ is zero, except for Case 4, where $A_{sc} = 1350 \text{ mm}^2$. The simplified method is not strictly applicable to Case 4. Values of $\gamma_1$ and $\gamma_2$ were determined from Equations 76 and 84, and substituted into Equation 83 to obtain the values of $\Delta \kappa_c^*$ given in Table 3. For comparison purposes, values of $\Delta \kappa_c^*$ obtained from the more accurate analysis in Example 5 are given in the last column of Table 3. For Case 3 the result is within 20 per cent. For the other cases the results are within about 15 per cent, even for Case 4 where the cross-section contains compressive reinforcement.

**TABLE 3.**

<table>
<thead>
<tr>
<th>Case</th>
<th>$A_{st}$ (mm$^2$)</th>
<th>$A_{sc}$ (mm$^2$)</th>
<th>$\Delta \kappa_c^*$ (mm$^{-1}$)</th>
<th>(from Table 1) $\Delta \kappa_c^*$ (mm$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2700</td>
<td>0</td>
<td>2.26E-7</td>
<td>1.95E-7</td>
</tr>
<tr>
<td>2</td>
<td>1350</td>
<td>0</td>
<td>1.46E-7</td>
<td>1.26E-7</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>5.96E-8</td>
<td>4.9E-8</td>
</tr>
<tr>
<td>4</td>
<td>1350</td>
<td>1350</td>
<td>1.46E-7</td>
<td>1.21E-7</td>
</tr>
</tbody>
</table>

8. Simplified expression for shrinkage curvature, $\kappa_{sh}^*$

A similar approach can be used to obtain a closed form, simplified expression for shrinkage warping. We consider a general concrete section, for example of I or T shape and with total concrete area $A_c$ and overall depth $D$. The depth to the centroid of the concrete section from the top fibre is $d_c$. We allow the concrete to shrink freely, i.e. it is assumed not to be bonded to the steel or tendon. There is uniform shortening of the concrete as a result of the strain $e_{cs}$, but there is no change in strain in the steel or tendon.
To simplify the analysis we replace the reinforcing steel and the tendon by an equivalent steel area $A_{eq}$ located at depth $d_{eq}$ below the top surface of the section, as used previously in dealing with creep. Equations 85 and 86 give expressions for these quantities. In fact, because the shrinkage strain increment is uniform over the section, compressive steel reinforcement can also be included in this simplified analysis of shrinkage. We now apply a resultant compressive force $\Delta X_p$ to the equivalent steel area at depth $d_{eq}$ and an equal tensile force $\Delta X_c$ to the concrete at the same level. The magnitude of $\Delta X_p$ is chosen so as to restore compatibility of the concrete strain and the steel strain increment at depth $d_{eq}$. The compressive strain increment in the steel, $\Delta \varepsilon_p$, plus the tensile strain increment in the concrete at depth $d_{eq}$, $\Delta \varepsilon_{c_{es}}$, add up to $\varepsilon_{cs}$:

$$\Delta \varepsilon_p + \Delta \varepsilon_{c_{es}} = \varepsilon_{cs} \quad \text{(EQ 87)}$$

When the analysis is carried out it is found that the force has the value

$$\Delta X_p = \gamma_1 A_c E_c \varepsilon_{cs} \quad \text{(EQ 88)}$$

The parameter $\gamma_1$ has already appeared in the creep analysis in Section 7, and is given by Equation 76. By substituting expressions for the increments in strain in the top and bottom fibres, $\Delta \varepsilon_{ct}$ and $\Delta \varepsilon_{c_{es}}$, into the shrinkage curvature $(\Delta \varepsilon_{ct} - \Delta \varepsilon_{c_{es}})/D$, we obtain the following approximation for the shrinkage curvature in the section:

$$\kappa^\ast_{sh} = -\gamma_1 \gamma_2 \left( \frac{\varepsilon_{cs}}{D} \right) \quad \text{(EQ 89)}$$

where $D$ is the overall depth of the section.

The shrinkage curvature is positive because the equivalent steel area is in the lower part of the concrete section. The parameters $\gamma_2$ and $\gamma_1$ are non-dimensional functions of the section properties and are the same as those that appeared in the creep analysis in Section 7. We rewrite them here, in terms of the equivalent steel area:

$$\gamma_1 = \frac{1}{1 + \frac{1}{n_{peq}} + \frac{A_{eq}^2}{I_c}} \quad \text{(EQ 90)}$$

$$\gamma_2 = \frac{A_c e_{eq} D}{I_c} \quad \text{(EQ 91)}$$

For conditions after all shrinkage has occurred, $t = t^*$, it is only necessary to substitute the final value of $\varepsilon_{cs}$ in Equation 89, together with values of the $\gamma$ terms obtained from Equations 90 and 91.

**EXAMPLE 8: SIMPLIFIED SHRINKAGE WARPING CALCULATION**

To provide an example of the simplified method for evaluating shrinkage warping, Cases (a), (b) and (c) from Example 6 and Table 2 are now recalculated. Cross-sectional data are as for the previous cases. Values of the section parameters $\gamma_1$ and $\gamma_2$ were obtained using Equations 90 and 91 and substituted into Equation 89 to determine $\kappa^\ast_{sh}$. The results given
in Table 4 are very close to the values obtained previously. Differences are mainly due to rounding errors.

<table>
<thead>
<tr>
<th>Case</th>
<th>$A_{st}$ (mm$^2$)</th>
<th>$A_{sc}$ (mm$^2$)</th>
<th>$\kappa^*_{sh}$ (mm$^{-1}$)</th>
<th>$\kappa^*_{sh}$ (mm$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>2700</td>
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<td>1.93E-7</td>
<td>1.96E-7</td>
</tr>
<tr>
<td>(b)</td>
<td>1350</td>
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<td>1.26E-7</td>
<td>1.28E-7</td>
</tr>
<tr>
<td>(c)</td>
<td>0</td>
<td>0</td>
<td>0.51E-7</td>
<td>0.051E-7</td>
</tr>
</tbody>
</table>

9. Acknowledgement

This report was originally prepared as an appendix for a new edition of a textbook on prestressed concrete design. I wish to express my thanks for help received from my colleague and friend, K A Faulkes, who provided critical comments on early drafts and undertook various calculations which have been incorporated in the examples.

10. References


(a) Concrete section with axial prestress

(b) Concrete section with symmetric reinforcement and axial prestress

(c) Concrete section with eccentric prestress

(d) Concrete section with eccentric prestress and reinforcement

Figure 1: Prestressed sections for analysis of creep and shrinkage
Figure 2: One-step analysis of creep, axially prestressed member

AA': prior to prestressing, $t_o$
BB': just after prestressing, $t_o$
CC': after free creep, $t^*$
DD': after restoration of compatibility, $t^*$
Figure 3: Three-step analysis of creep, axially prestressed member
Figure 4: Creep analysis for a prestressed, reinforced beam section
Figure 5: Shrinkage warping for a prestressed, reinforced section